# CHAPTER-1

## HEAT, WAVE AND LAPLACE EQUATIONS

**Structure**



### 1.1 Introduction

In this section, the temperature distribution is studied in several cases. For finding the temperature distribution we require to solve the Heat equation with different Boundary Value Problem (B.V.P.), whereas to find the steady state temperature distribution we require to attempt a solution of Laplace equation and to obtain motion of vibrating string we find a solution of Wave equation.

#### 1.1.1 Objective

The objective of these content is to provide some important results to the reader like:

**(i)** Temperature distribution in a bar with ends at zero temperature, insulated ends, radiating ends and ends at different temperature.
**(ii)** Steady state Temperature distribution in a finite, semi-infinite and infinite plate
**(iii)** Heat conduction in semi-infinite and infinite bar
**(iv)** Solution of Heat, Laplace and Wave equation in various cases

### 1.2. Method of Separation of Variables to solve B.V.P. associated with One Dimensional Heat Equation

A parabolic equation of the type

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad \text{---(1)}$$

k being a dissasivity (constant) and $u(x,t)$ being temperature at a point $(x,t)$ of a solid at time t is known as Heat Equation in one dimension.

We now proceed to discuss the method of separation of variables to solve B.V.P., with boundary conditions:

$$u(0,t)=0 \text{ and } u(l,t)=0 \qquad ...(2)$$

and

$$u(x,0)= f(x) \text{ and } \left[\frac{\partial u}{\partial t}\right]_{t=0} = v(x) \qquad ...(3)$$

Suppose the solution of (1) is

$$u(x,t)=X(x)T(t) \qquad ...(4)$$

where X(x) is a function of x only and T(t) is a function of t only.
Therefore, we have

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}T(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2X}{dx^2}T(t) \qquad ...(5)$$

and

$$\frac{\partial u}{\partial t} = X(x)\frac{dT}{dt}$$

Inserting (5) into (1), we obtain

$$T(t)\frac{d^2X}{dx^2} = \frac{1}{k}X(x)\frac{dT}{dt}$$

Dividing both sides by u(x,t)=X(x)T(t),we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{kT}\frac{dT}{dt} \qquad ...(6)$$

Now, L.H.S. of (6) is independent of t and R.H.S. is independent of x, either side of (6) can be equated to some constant of separation. If constant of separation is $p^2$, then

$$\frac{1}{X}\frac{d^2X}{dx^2} = p^2 \quad \text{and} \quad \frac{1}{kT}\frac{dT}{dt} = p^2$$

$$\text{or } \frac{d^2X}{dx^2} - p^2X = 0 \qquad ...(7)$$

$$\textit{and } \frac{dT}{dt} - p^2kT = 0 \qquad ...(8)$$

These equations have the solutions

$$X(x)=c_1e^{px}+c_2e^{-px} \text{ and } \mathrm{T}(t)=Ae^{kp^2t} \qquad ...(9)$$

In view of (2), (4) implies

$$u(0,t)=0 \Rightarrow \mathrm{X}(0)T(t)=0$$

Here, either X(0)=0 or T(t)=0. If T(t) is assumed to be zero identically then u(x,t)=X(x)T(t) is zero identically, that is the temperature function is zero identically, which is of no interest. Thus, we take

X(0)=0

Similarly, $u(l,t)=0 \Rightarrow X(l)T(t)=0 \Rightarrow X(l)=0$

Thus, we have

$$X(0)=X(l)=0 \qquad ...(10)$$

Now, applying (10) on (9), we get

$$c_1+c_2=0 \text{ and } c_1e^{pl}+c_2e^{-pl}=0$$

This system has a trivial solution

$$c_1=c_2=0$$

and so X(x)=0, then the temperature function becomes zero which is not being assumed.

Now, let $p^2=0$, then $(7)$ and $(8)$ implies

$$\frac{d^2X}{dx^2}=0 \text{ and } \frac{dT}{dt}=0$$

$$\Rightarrow X(x)=c_1x+c_2 \text{ and } T(t)=c \qquad ...(11)$$

Now, applying (10) on (11), we obtain:

$$c_1=c_2=0$$

$$\Rightarrow X(x)=0$$

Again, the temperature function becomes zero and is of no interest.

So, assume that the constant of separation is $-p^2$, so that

$$\frac{d^2X}{dx^2}+p^2X=0 \qquad ...(12)$$

$$\frac{dT}{dt}+kp^2T=0 \qquad ...(13)$$

Solution of (12) is

$$X(x)=c_1\cos px+c_2\sin px \qquad ...(14)$$

In view of (10), (14) implies

$$X(0)=c_1=0 \text{ and}$$

$$X(l)=c_2\sin pl=0$$

$$\Rightarrow pl=n\pi \text{ for } n\neq 0, \text{ n being an integer.}$$

$$\Rightarrow p=\frac{n\pi}{l}$$

For n≠0, we have infinite many solutions

$$X_n(x) = a_n \sin\frac{n\pi x}{l} \text{ ; n =1,2,. ..} \qquad ...(15)$$

Now, for $p = \dfrac{n\pi}{l}, (13)$ gives

$$\frac{dT}{dt} + k\left(\frac{n\pi}{l}\right)^2 T = 0$$

$$or \ \frac{dT}{dt} + \lambda_n T = 0, \text{ where } \lambda_n = \frac{kn^2\pi^2}{l^2}$$

Its general solution is

$$T(t) = c_n e^{-\lambda_n t} \qquad ...(16)$$

Combining (15) and (16), we have

$$u_n(x,t) = c_n e^{-\lambda_n t} a_n \sin\frac{n\pi x}{l} \qquad ...(17)$$

where n =1,2,…

Now, for the general solution, we have

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin\frac{n\pi x}{l} \qquad ...(18)$$

$$where \ \mathrm{b}_n = a_n c_n$$

giving

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = f(x) \qquad ...(19)$$

$$and \left[\frac{\partial u}{\partial t}\right]_{t=0} = \left[\sum_{n=1}^{\infty}\left\{-\lambda_n b_n \sin\frac{n\pi x}{l}.e^{-\lambda_n t}\right\}\right]_{t=0}$$

$$= -\sum_{n=1}^{\infty} \lambda_n b_n \sin\frac{n\pi x}{l} = v(x) \qquad ...(20)$$

From (19) and (20), the constant $b_n$ can be determined easily and thus, (18) represents the solution of Heat equation.

### 1.2.1 Ends of the Bar Kept at Temperature Zero

Suppose we want the temperature distribution u(x,t) in a thin, homogeneous bar of length L, given that the initial temperature in the bar at time zero in the section at perpendicular to the x-axis is specified by u(x,0)=f(x). The ends of the bar are maintained at temperature zero for all time. The boundary value problem modeling this temperature distribution is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (0 < x < L, t > 0) \qquad ...(1)$$

$$u(0,t) = u(L,t) = 0 \quad (t > 0) \qquad ...(2)$$

$$u(x,0) = f(x) \quad (0 < x < L) \qquad ...(3)$$

Put $u(x,t) = X(x)T(t) \qquad ...(4)$

into the equation (1) to get

$$XT' = a^2 X''T \qquad ...(5)$$

where primes denote differentiation w.r.t. the variable of the function.

Then, $\dfrac{X''(x)}{X(x)} = \dfrac{T'(t)}{a^2 T(t)} \qquad ...(6)$

The R.H.S. of this equation is a function of t only and L.H.S. a function of x only and these variables are independent. We could, e.g. choose any t, we like, thereby fixing the right side of the equation at a constant value. The left side would then have to equal this constant for all x. We therefore, conclude that $\frac{X''}{X}$ is constant. But then $\frac{T'}{a^2T}$ must equal the same constant, which we will designate $-\lambda$ (The negative sign is a convention; we would eventually get the same solution if we used $\lambda$ ). The constant $\lambda$ is called the separation constant.

Thus, we have

$$\frac{X''}{X} = \frac{T'}{a^2T} = -\lambda$$

giving us two ordinary differential equations

$$X'' + \lambda X = 0$$

$$T' + \lambda a^2 T = 0$$

Now consider the boundary conditions. First

$$u(0,t) = X(0)T(t) = 0$$

$$\Rightarrow X(0) = 0 \text{ or } \mathrm{T}(t) = 0$$

If $\mathrm{T}(t) = 0$ for all t, then the temperature in the bar is always zero. This is indeed the solution if f(x)=0. Otherwise, we must assume that T(t) is non-zero for some t and conclude that

$$X(0) = 0$$

Similarly,

$$u(L,t) = X(L)T(t) = 0$$

$$\Rightarrow X(L) = 0$$

We now have the following problems for X and T

$$X''+\lambda X=0$$
$$X(0)=X(L)=0$$
and $$T'+\lambda a^2 T=0$$

We will solve for $X(x)$ first because we have the most information about X. The problem is a regular Strum-Liouville Problem on [0,L]. A value for $\lambda$ for which the problem has a non-trivial solution is called an eigen value of this problem. For such a $\lambda$, any non-trivial solution for X is called an eigen function.

**Case 1:** $\lambda=0$

Then, $X''=0$, so $X(x)=cx+d$, Now $X(0)=d=0$, so $X(x)=cx$. But then $X(L)=cL=0 \Rightarrow c=0$

Thus, there is only the trivial solution for this case. We conclude that 0 is not an eigen value of problem.

**Case 2:** $\lambda<0$

Write $\lambda=-k^2$, with k >0. Then, equation for X(x) is

$$X''-k^2X=0$$

with general solution

$$X(x)=ce^{kx}+de^{-kx}$$

Now, $X(0)=c+d=0 \Rightarrow c=-d$

Therefore, $X(x)=ce^{kx}-ce^{-kx}=c\left(e^{kx}-e^{-kx}\right)$

Next, $X(L)=c\left(e^{kL}-e^{-kL}\right)=0$

Here, $e^{kL}-e^{-kL}\neq 0$, because kL>0, so c=0. Therefore, there are no nontrivial solutions of the problems if $\lambda<0$, and this problem has no negative eigen value.

**Case 3:** $\lambda>0$

Write $\lambda=k^2$, with k>0. The general solution of

$$X''+k^2X=0$$

is $X(x)=c\cos kx+d\sin kx$

Now, $X(0)=c=0$, so $X(x)=d\sin kx$.

Therefore, $X(L)=d\sin kL=0$

To have a non-trivial solution, we must be able to choose $d\neq 0$.

This require that $\sin kL = 0$, which occurs if $kL$ is a positive integer multiple of $\pi$,

say $kL = n\pi$ .

Thus, choose $k = \frac{n\pi}{L}$, for n=1,2,…

For each such n, we can choose

$$X_n(x) = d_n \sin\left(\frac{n\pi x}{L}\right)$$

This is a eigen function of the given problem corresponding to the eigen value $\lambda = k^2 = \frac{n^2\pi^2}{L^2}$

Now, return to the problem for T with $\lambda = \frac{n^2\pi^2}{L^2}$, the differential equation is

$$T' + \frac{n^2\pi^2a^2T}{L^2} = 0$$

with general solution

$$T_n(t) = a_n e^{\frac{-n^2\pi^2a^2t}{L^2}}$$

For each positive integer n, we can get

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2a^2t}{L^2}}, \text{ where } c_n = a_n d_n$$

This function satisfies the heat equation and the boundary conditions $u(0,t) = u(L,t) = 0$ on $t \geq 0$ To satisfy the initial condition for a given n, however, we need

$$u_n(x,0) = c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

And this is possible only if $f(x)$ is a constant multiple of this sine function. Usually, to satisfy the initial condition we must attempt a superposition of all the $u_n\,'s$:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2a^2t}{L^2}}$$

The initial condition now requires that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

Which we recognize as the Fourier sine expansion of $f(x)$ on [0.L]. Therefore, choose the $c_n$'s as the Fourier sine coefficients of $f(x)$ on [0,L]:

$$c_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi x}{L}\right)d\xi$$

With certain conditions on $f(x)$ this Fourier sine series converges to $f(x)$ for $0 < x < L$ and the formal solution of the boundary value problem is

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right)\sin\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2a^2t}{L^2}}$$

**Example:** As a special example, suppose the bar is kept at constant temperature A, except at its ends, which are kept at temperature zero. Then,

$$f(x) = A \qquad (0 < x < L)$$

and

$$c_n = \frac{2}{L}\int_0^L A\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2A}{n\pi}(1-\cos n\pi)$$
$$= \frac{2A}{n\pi}\left(1-(-1)^n\right)$$

The solution in this case is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2A}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2a^2t}{L^2}}$$
$$= \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi x}{L}\right)e^{\frac{-(2n-1)^2\pi^2a^2t}{L^2}}$$

We got the last summation from the preceding line by noticing that $1-(-1)^n = 0$ if n is even, so all the terms in the series vanish for n even and we need only retain the terms with n odd. This is done by replacing $n$ with $2n-1$, there by summing over only the odd positive integers.

**Problems:** Solve the following boundary value problem:

1. $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L,\ t > 0)$

$u(0,t) = u(L,t) = 0 \qquad (t > 0)$

$u(x,0) = x(L-x) \qquad (0 < x < L)$

2. $$\frac{\partial u}{\partial t}=4\frac{\partial^2 u}{\partial x^2} \qquad (0<x<L, t>0)$$
$$u(0,t)=u(L,t)=0 \qquad (t>0)$$
$$u(x,0)=x^2(L-x) \qquad (0<x<L)$$

3. $$\frac{\partial u}{\partial t}=3\frac{\partial^2 u}{\partial x^2} \qquad (0<x<L, t>0)$$
$$u(0,t)=u(L,t)=0 \qquad (t>0)$$
$$u(x,0)=L\left[1-\cos\left(\frac{2\pi x}{L}\right)\right] \qquad (0<x<L)$$

### 1.2.2 Temperature in a Bar with Insulated Ends

Consider heat conduction in a bar with insulated ends, hence no energy loss across the ends. If the initial temperature is given by $f(x)$, then the temperature function is modeled by the B.V.P.

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \qquad (0<x<L, t>0)$$
$$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(L,t)=0 \qquad (t>0)$$
$$u(x,0)=f(x) \qquad (0<x<L)$$

We will solve for $u(x,t)$, leaving out some details, which are the same as in the preceding problem. Set

$$u(x,t)=X(x)T(t)$$

And substitute into the heat equation to get

$$\frac{X''}{X}=\frac{T'}{a^2T}=-\lambda$$

In which $\lambda$ is the separation constant. Then,

$$X''+\lambda X=0$$

and $T'+\lambda a^2 T=0$

as before. Also,

$$\frac{\partial u}{\partial x}(0,t)=X'(0)T(t)=0$$

implies that $X'(0)=0$. The other boundary condition implies that $X'(L)=0$. The other boundary condition implies that $X'(L)=0$. The problem for X is therefore

$$X''+\lambda X=0 \qquad ...(1)$$
$$X'(0)=X'(L)=0 \qquad ...(2)$$

We seek values of $\lambda$ for which this problem has non-trivial solutions.

Consider cases on $\lambda$:

**Case 1:** $\lambda=0$

The general solution for (1) is

$$X(x)=cx+d$$

Since $X'(0)=0=c$, therefore, 0 is an eigen value of (1) with eigen function.

$$X(x)=\text{constant}\neq 0$$

**Case 2:** $\lambda<0$

Write $\lambda=-k^2$ with $k>0$. Then, $X''-k^2X=0$, with general solution

$$X(x)=ce^{kx}+de^{-kx}$$

Now,

$$X'(0)=kc-kd=0\Rightarrow c=d \quad [\because k>0]$$
$$\therefore X(x)=c\left(e^{kx}+e^{-kx}\right)$$

Next,

$$X'(L)=ck\left(e^{kL}-e^{-kL}\right)=0$$

This is zero only if c=0. But this forces $X(x)=0$, so choosing $\lambda$ negative eigen value.

**Case 3:** $\lambda>0$

Set $\lambda=k^2$, with $k>0$.Then,

$$X''+k^2X=0$$

with general solution

$$X(x)=c\cos kx+d\sin kx$$

Now, $X'(0)=dk=0$

implies that d=0. Then, $X(x)=c\cos kx$.

Next, $X'(L)=-ck\sin kL=0$

In order to get a non-trivial solution, we need $c \neq 0$, and must choose k so that $\sin kL = 0$, therefore $kL = n\pi$

for n, a positive integer, and this problem has eigen values

$$\lambda = k^2 = \frac{n^2\pi^2}{L^2}\text{; for n=1,2,…}$$

Corresponding to such an eigen value, the eigen function is

$$X_n(x) = c_n \cos\left(\frac{n\pi x}{L}\right)\text{, for n=1,2,…}$$

We can combine case 1 and case 3, by writing the eigen values as

$$\lambda = \frac{n^2\pi^2}{L^2} \quad \text{for n=0,1,2,...}$$

and eigen functions as

$$X_n(x) = c_n \cos\left(\frac{n\pi x}{L}\right)$$

This is a constant functions, corresponding to $\lambda = 0$, when n=0.

The equation for T is

$$T' + \frac{n^2\pi^2a^2T}{L^2} = 0$$

When n=0, this has solutions

$$T_0(t) = \text{constant} = d_0$$

If n=1,2,…, then

$$T_n(t) = d_n e^{\frac{-n^2\pi^2a^2T}{L^2}}$$

Now let

$$u_0(x,t) = \text{constant} = a_0$$

and $u_n(x,t) = a_n \cos\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2a^2t}{L^2}}$, where $a_n = c_n d_n$

Each of these functions satisfies the heat equation and boundary conditions. To satisfy the initial condition, we must usually attempt a superposition of these functions:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2a^2t}{L^2}}$$

We must choose the $a_n$'s so that

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

This is a Fourier cosine expansion of $f(x)$ on [0,L], so choose

$$a_0 = \frac{1}{L}\int_0^L f(\xi)\,d\xi$$

and, for n=1,2,…

$$a_n = \frac{2}{L}\int_0^L f(\xi)\cos\left(\frac{n\pi\xi}{L}\right)d\xi$$

The solution is

$$u(x,t) = \frac{1}{L}\int_0^L f(\xi)\,d\xi + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\cos\left(\frac{n\pi\xi}{L}\right)d\xi\right)\cos\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2a^2t}{L^2}}$$

**Example:** Suppose the left half of the bar is initially at temperature *A* and the right half at temperature zero. Then,

$$f(x) = \begin{cases} A & ,\ 0<x<\dfrac{L}{2} \\ 0 & ,\ \dfrac{L}{2} < x < L \end{cases}$$

$$\therefore \quad a_0 = \frac{1}{L}\int_0^{\frac{L}{2}} A\,d\xi = \frac{A}{2}$$

$$\text{and}\ \ a_n = \frac{2}{L}\int_0^{\frac{L}{2}} A\cos\left(\frac{n\pi\xi}{L}\right)d\xi = \frac{2A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

The solution for this temperature distribution is

$$u(x,t) = \frac{A}{2} + \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2a^2t}{L^2}}$$

Since $\sin\left(\frac{n\pi}{2}\right)$ is zero if n is even and equals $(-1)^{k+1}$ if n=2k+1. We may omit all terms of this series in which the summation index is even, and sum over only the odd positive integers. This is done by replacing n with 2n-1 in the function being summed. Then,

$$u(x,t) = \frac{A}{2} + \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos\left(\frac{(2n-1)\pi x}{L}\right)e^{\frac{-(2n-1)^2\pi^2a^2t}{L^2}}$$

**Problems:**

Solve the following B.V.P.'s:

1. $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad (0<x<\pi, t>0)$

$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(\pi,t)=0 \qquad (t>0)$

$u(x,0)=\sin x \qquad (0<x<\pi)$

2. $\frac{\partial u}{\partial t}=4\frac{\partial^2 u}{\partial x^2} \qquad (0<x<2\pi, t>0)$

$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(2\pi,t)=0 \qquad (t>0)$

$u(x,0)=x(2\pi-x) \qquad (0<x<2\pi)$

3. A thin homogeneous bar of length L has insulated ends initial temperature B, a positive constant. Find the temperature distribution in the bar.

4. A thin homogeneous bar of length L has initial temperature equal to a constant B and the right end (x=L) is insulated, while the left end is kept at a zero temperature. Find the temperature distribution in the bar.

5. A thin homogeneous bar of thermal diffusivity 9 and length 2 cm and insulated has its left end maintained at temperature zero, while the right end is perfectly insulated. The bar has an initial temperature given by $f(x)=x^2$ for 0<x<2. Determine the temperature distribution in the bar. What is $\lim_{t\to\infty} u(x,t)$?

### 1.2.3 Temperature Distribution in a Bar with Radiating End

Consider a thin, homogeneous bar of length *L*, with the left end maintained at temperature zero, while the right end radiates energy into the surrounding medium, which also is kept at temperature zero. If the initial temperature in the bar's cross section at *x* is *f(x)*, then the temperature distribution is modeled by the B.V.P.

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \qquad (0\le x\le L, t>0)$$

$$u(0,t)=0,\quad \frac{\partial u}{\partial x}(L,t)=-Au(L,t) \qquad (t\ge 0)$$

$$u(x,0)=f(x) \qquad (0\le x\le L)$$

The boundary condition at L assumes that heat energy radiates from this end at a rate proportional to the temperature at that end of the bar, A is a positive constant called the transfer co-efficient.

Let $u(x,t)=X(x)T(t)$ to obtain, as before,

$$X''+\lambda X=0$$

$$T'+\lambda a^2 T=0$$

Since,

$$u(0,t) = X(0)T(t) = 0, \text{ then}$$
$$X(0) = 0$$

as $T(t) = 0$, implies that $u(x,t) = 0$ which is possible only if $f(x) = 0$. The condition at the right end of the bar implies that

$$X'(L)T(t) = -AX(L)T(t)$$
$$\Rightarrow X'(L) + AX(L) = 0$$

The problem for $X(x)$ is therefore,

$$X'' + \lambda X = 0$$
$$X(0) = X'(L) + AX(L) = 0$$

From the strum-Liouville theorem, we can be confident that this problem has infinitely many eigen values $\lambda_1, \lambda_2, \ldots,$ each of which is associated with a non-trivial solution, or eigen functions, $X_n(x)$. We would like, however, to know these solutions, so we will consider cases:

**Cases 1:** $\lambda = 0$,

Then, the solution for $X(x)$ is

$$X(x) = cx + d$$

Since, $X(0) = 0 = d$, then

$$X(x) = cx$$

But then

$$X'(x) = c = -AX(L) = -AcL$$

Then,

$$c(1 + AL) = 0$$

But $1 + AL > 0$, so c=0 and we get only the trivial solution from this case. This means that 0 is not an eigen value of this problem.

**Case 2:** $\lambda < 0$, write $\lambda = -k^2$, with $k > 0$ .Then,

$$X'' - k^2 X = 0, \text{ so}$$

$$X(x) = ce^{kx} + de^{-kx}$$

Now, $X(0) = c + d = 0 \Rightarrow d = -c.$

$$\therefore X(x) = c\left(e^{kx} - e^{-kx}\right) = 2c\sinh(kx)$$

Then, $X'(L) = 2ck\cosh(kL) = -Ac\sinh(kL)$

To have a non-trivial solution, we must have $c \neq 0$ and this requires that

$$2k\cosh(kL) + A\sinh(kL) = 0$$

This is impossible because $Lk > 0,$ so the left side of this equation is a sum of positive numbers. Therefore, this problem has no negative eigen value.

**Case 3:** $\lambda > 0,$ write $\lambda = k^2,$ with $k > 0$. Then,

$$X'' + k^2 X = 0, \text{ so}$$

$$X(x) = c\cos kx + d\sin kx$$

Now, $X(0) = c = 0,$ so $X(x) = d\sin kx.$

Further, $X'(L) + AX(L) = dk\cos(kL) + Ad\sin(kL) = 0$

To have a non-trivial solution, we must have $d \neq 0$, and this requires that

$$k\cos(kL) + A\sin(kL) = 0$$

or $\tan(kL) = \dfrac{-k}{A}$

Let $z = kL$. Then, this equation is

$$\tan(z) = \frac{-z}{AL}$$

Since $k = \dfrac{z}{L},$ then $\lambda_n = \dfrac{z_n^2}{L^2}$

is an eigen value of this problem for each positive integer n which is shown in Figure below,

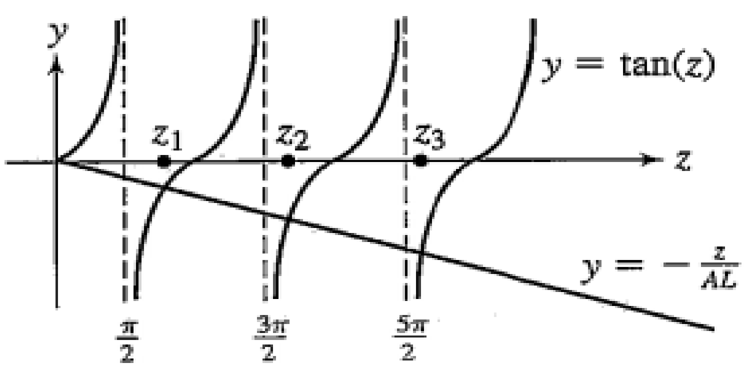


Figure: The eigen values of the problem for a bar with radiating ends with corresponding eigen function

$$X_n(x) = a_n \sin\left(\frac{z_n x}{L}\right)$$

The equation for T is

$$T' + \frac{a^2 z_n^2 T}{L^2} = 0$$

So $T_n(t) = d_n e^{\frac{-a^2 z_n^2 t}{L^2}}$

For each positive integer n, let

$$u_n(x,t) = c_n \sin\left(\frac{z_n x}{L}\right) e^{\frac{-a^2 z_n^2 t}{L^2}} \text{ where } c_n = a_n d_n$$

Each such function satisfies the heat equation and the boundary conditions. To satisfy the initial conditions, let

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{z_n x}{L}\right) e^{\frac{-a_n z_n^2 t}{L^2}}$$

we must choose the $c_n$'$s$ so that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{z_n x}{L}\right) = f(x)$$

Unlike what we encountered in the other two examples, this is not a standard's Fourier series, because of the $z_n$'$s$. Indeed, we do not know these numbers, because they are solutions of a transcendental equation we cannot solve exactly.

At this point we must rely on the Strum- Liouville theorem, which states that the eigen functions of the Strum- Liouville problem are orthogonal on [0,L] with weight function 1. This means that if n and m are distinct positive integers, then

$$\int_0^L \sin\left(\frac{z_m x}{L}\right)\sin\left(\frac{z_n x}{L}\right) dx = 0$$

This is like the orthogonality relationship used to derive co-efficient of Fourier series and can be exploited in the same way to find the

$$c_n = \frac{\int_0^L f(x)\sin\left(\frac{z_n x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{z_n x}{L}\right) dx}$$

With this choice of co-efficient, the solution is

$$u(x,t)=\sum_{n=1}^{\infty}\left[\frac{\int_0^L f(\xi)\sin\left(\frac{z_n\xi}{L}\right)d\xi}{\int_0^L \sin^2\left(\frac{z_n\xi}{L}\right)d\xi}\right]\sin\left(\frac{z_n x}{L}\right)e^{\frac{-a^2z_n^2t}{L^2}}$$

**Problems:**

1. A thin, homogeneous bar of thermal diffusivity 4 and length 6 cm with insulated sides, has its end maintained at temperature zero. Its right end is radiating (with transfer co-efficient $\frac{1}{2}$) into the surrounding medium, which has temperature zero. The bar has an initial temperature given by $f(x)=x(6-x)$. Approximate the temperature distribution $u(x,t)$ by finding the fourth partial sum of the series representation for $u(x,t)$.

### 1.2.4 Heat Conduction in a Bar with Ends at Different Temperature

Consider a thin, homogeneous bar extending from $x=0$ to $x=L$. The left end is maintained at constant temperature $T_1$ and the right end at constant temperature $T_2$. The initial temperature throughout the bar in the cross-section at $x$ is $f(x)$.

The boundary value problem for the temperature distribution is:

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}\ (0<x<L\ ,\ t>0)$$
$$u(0,t)=T_1\ ,\ u(L,t)=T_2\ (t>0)$$
$$u(x,0)=f(x)\ (0\le x\le L)$$

Put $u(x,t)=X(x)T(t)$ into the heat equation to obtain,

$$X''+\lambda X=0$$
$$T'+\lambda a^2T=0$$

Unlike the preceding example, there is nothing in this partial differential equation that prevents separation of the variables. The difficulty encountered here is with the boundary conditions which are non-homogeneous ($u(0,t)$ and $u(L,t)$ may be non-zero). To see the effect of this consider, $u(0,t)=X(0)T(t)=T_1$

If $T_1=0$, we could conclude that $X(0)=0$. But if $T_1\neq 0$, this equation forces us to conclude that $T(t)=\frac{T_1}{X(0)}=\text{constant}$.This is a condition, we cannot except to satisfy. The boundary condition at $L$ possess the same problem.

We attempt to eliminate the problem by perturbing the function. Set

$$u(x,t)=U(x,t)+\psi(x)$$

We want to choose $\psi(x)$ to obtain a problem, we can solve.

Substitute $u(x,t)$ into the partial differential equation to get

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + a^2 \psi''(x)$$

We obtain the heat equation for U if $\psi''(x) = 0$. Integrating twice, $\psi(x)$ must have the form

$$\psi(x) = Cx + D \qquad ...(1)$$

Now, consider the boundary conditions, first

$$u(0,t) = T_1 = U(0,t) + \psi(0)$$

This condition becomes $U(0,t) = 0$ if we choose $\psi(x)$ so that

$$\psi(0) = T_1 \qquad ...(2)$$

The condition

$$u(L,t) = T_2 = U(L,t) + \psi(L)$$

becomes $U(L,t) = 0$ if

$$\psi(L) = T_2 \qquad ...(3)$$

Now, use (2) and (3) to solve for C and D in (1),

$$\psi(0) = D = T_1$$

and $\psi(L) = CL + T_1 = T_2 \Rightarrow C = \frac{1}{L}(T_2 - T_1)$

Thus, choose

$$\psi(x) = \frac{1}{L}(T_2 - T_1)x + T_1$$

with this choice, the boundary value problem for $U(x,t)$ is

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2}$$

$$U(0,t) = U(L,t) = 0$$

$$U(x,0) = u(x,0) - \psi(x) = f(x) - \frac{1}{L}(T_2 - T_1)x - T_1$$

We have solved this problem earlier, with the solution

$$U(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L \left[ f(\xi) - \frac{1}{L}(T_2 - T_1)x + T \right] \sin\left(\frac{n\pi\xi}{L}\right) d\xi \right) \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2 a^2 t}{L^2}}$$

Once, we know this function, then

$$u(x,t) = U(x,t) + \frac{1}{L}(T_2 - T_1)x + T_1$$

### 1.3 Steady–State Temperature in Plates

The two-dimensional Heat equation is

$$\frac{\partial u}{\partial t} = a^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = a^2\nabla^2 u$$

The steady-state case occurs when we set $\frac{\partial u}{\partial t} = 0$. In this event, the Heat equation is Laplace's equation

$\nabla^2 u = 0$

A Dirichlet problem consists of Laplace's equation, to be solved for (x,y) in a region R of the plane, together with prescribed values the solution is to assumes on the boundary of R, which is usually a piecewise smooth curve. If we think of R as a flat plate, then we are finding the steady-state temperature distribution throughout a plate, given the temperature at all timers on its boundary.

#### 1.3.1 Steady-State Temperature in a Rectangular Plate

Consider a flat rectangular plate occupying the region R in the xy-plane by $0 \le x \le a,\ 0 \le y \le b.$ Suppose the right side is kept at constant temperature T, while the other sides are kept at temperature zero. The boundary value problem for the steady-state temperature distribution is:

$$\begin{aligned}
&\nabla^2 u = 0 \ (0 < x < a, 0 < y < b) \\
&u(x,0) = u(x,b) = 0 \ (0 < x < a) \\
&u(0,y) = 0 \quad (0 < y < b) \\
&u(a,y) = T \quad (0 < y < b)
\end{aligned}$$

Put $u(x,y) = X(x)Y(y)$ into Laplace's equation to get

$$\begin{aligned}
&X''Y + Y''X = 0 \\
&\frac{X''}{X} = -\frac{Y''}{Y}
\end{aligned}$$

Since the left side depends only on x and the right side only on y, and these variables are independent, both sides must equal the same constant.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda \text{ (say)}$$

Now, use the boundary condition:

$$\begin{aligned}
&u(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0 \\
&u(x,b) = X(x)Y(b) = 0 \Rightarrow Y(b) = 0
\end{aligned}$$

and $u(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0$

Therefore, $X(x)$ must satisfy

$$X'' - \lambda X = 0$$
$$X(0) = 0$$

and, Y must satisfy

$$Y'' + \lambda Y = 0$$
$$Y(0) = Y(b) = 0$$

This problem for $Y(y)$ was solved in the article (Ends of the bar kept at temperature zero) with $X(x)$ in place of $Y(y)$ and L in place of b.

The eigen values are

$$\lambda_n = \frac{n^2\pi^2}{b^2}$$

with corresponding eigen functions

$$Y_n(y) = b_n \sin\left(\frac{n\pi y}{b}\right) \text{ for n=1,2,...}$$

The problems for X is now

$$X'' - \frac{n^2\pi^2}{b^2} X = 0$$
$$X(0) = 0$$

The general solution of the differential equation is

$$X_n(x) = ce^{\frac{n\pi x}{b}} + de^{\frac{-n\pi x}{b}}$$

Since $X(0) = c + d = 0 \Rightarrow d = -c$ and so

$$X_n(x) = c\left[e^{\frac{n\pi x}{b}} - e^{\frac{-n\pi x}{b}}\right] = 2c_n \sinh\left(\frac{n\pi x}{b}\right)$$

For each positive integer n, let

$$u_n(x, y) = a_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right) \text{ ; where } a_n = 2b_n c_n$$

For each n and any choice of the constant $a_n$ this function satisfies Laplace's equation and the zero boundary conditions on three sides of the plate. For the non-zero boundary condition, we must use a superposition

$$u(a, y) = T = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{b}\right)\sinh\left(\frac{n\pi a}{b}\right)$$

This is a Fourier sine expansion of T on [a,b]. Therefore, choose the entire co-efficient

$\sin\left(\frac{n\pi y}{b}\right)$ as the Fourier sine co-efficient:

$$a_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b T\sin\left(\frac{n\pi y}{b}\right)dy$$
$$= \frac{2T}{b}\left[1-(-1)^n\right]\frac{b}{n\pi},$$

in which we have used the fact that $\cos n\pi = (-1)^n$, if n is an integer.

We now have

$$a_n = \frac{2T}{n\pi}\frac{1}{\sinh\left(\frac{n\pi a}{b}\right)}\left[1-(-1)^n\right]$$

The solution is

$$u(x,y) = \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\sinh\left(\frac{n\pi a}{b}\right)}\left[1-(-1)^n\right]\sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

As we have done before, observe that $1-(-1)^n$ equals 0 if n is even, and equals 2 if n is odd. We can therefore omit the even indices in this summation, writing the solution as:

$$u(x,y) = \frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh\left(\frac{(2n-1)\pi a}{b}\right)}\sinh\left(\frac{(2n-1)\pi x}{b}\right)\sin\left(\frac{(2n-1)\pi y}{b}\right)$$

**Problems:** 1. Solve for the steady-state temperature distribution in a flat plate covering the region $0 \le x \le a,\ 0 \le y \le b,$ if the temperature on the vertical sides and the bottom side are kept at zero while the temperature on the top side is a constant K.

2.Solve for the steady-state temperature distribution is a flat plate covering the region $0 \le x \le a,\ 0 \le y \le b,$ if the temperature on the left side is a constant $T_1$ and that on right side a constant $T_2$, while the top and bottom sides are kept at temperature zero.

[Hint: Consider two separate problems. In the first, the temperature on the left side is $T_1$ and the other sides are kept at temperature zero. In the second, the temperature on the right side is $T_2$, while the other sides are kept at zero. The sum of solutions of these problems is the solution of the original problem.]

**Remark:** It is possible to treat the case where the four sides are kept at different temperature (not necessarily constant), by considering four plates, in each of which the temperature is non-zero on only one side of the plate. The sum of the solutions of these four problems is the solution for the original plate.

### 1.3.2 Steady-State Temperature in a Circular Disc

Consider a thin disk of radius R, placed in the plane so that its centre is the origin. We will find the steady-state temperature distribution $u(r,\theta)$ as a function of polar co-ordinates. The Laplace's equation in polar co-ordinates is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

for $0 \le r \le R$ and $-\pi \le \theta \le \pi$

Assume that the temperature is known on the boundary of the disk:

$$u(R,\theta)=f(\theta) \qquad \text{for} \qquad -\pi \le \theta \le \pi$$

In order to determine a unique solution for u, we will specify two additional conditions, First we seek a bounded solution. This is certainly a physically reasonable condition. Second we assume periodically conditions:

$$u(r,\pi)=u(r,-\pi) \quad \text{and} \quad \frac{\partial u}{\partial \theta}(r,\pi)=\frac{\partial u}{\partial \theta}(r,-\pi)$$

These conditions account for the fact that $(r,\pi)$ and $(r,-\pi)$ are polar co-ordinates of the same point.

Attempt a solution

$$u(r,\theta)=F(r)G(\theta)$$

Substitute this into the Laplace's equation, we get

$$F''(r)G(\theta)+\frac{1}{r}F'(r)G(\theta)+\frac{1}{r^2}F(r)G''(\theta)=0$$

If $F(r)G(\theta)\neq 0$, this equation can be written

$$\frac{r^2F''(r)+rF'(r)}{F(r)}=-\frac{G''(\theta)}{G(\theta)}$$

Since the left side of this equation depends only on r and the right side only on $\theta$, and these variables are independent, both sides must equal same constant

$$\frac{r^2F''(r)+rF'(r)}{F(r)}=-\frac{G''(\theta)}{G(\theta)}=\lambda$$

which gives

$$r^2F''(r)+rF'(r)-\lambda F(r)=0 \qquad ...(1)$$

and $G''(\theta)+\lambda G(\theta)=0$

Now, consider the boundary conditions. First

$$u(r,\pi)=u(r,-\pi)\Rightarrow G(\pi)F(r)=G(-\pi)F(r)$$

Assuming $F(r)$ is not identically zero, then

$$G(\pi)=G(-\pi)$$

Similarly,

$$\frac{\partial u}{\partial\theta}(r,\pi)=F(r)G'(\pi)=\frac{\partial u}{\partial\theta}(r,-\pi)=F(r)G'(-\pi)$$
$$\Rightarrow G'(\pi)=G'(-\pi)$$

The problem to solve $G(\theta)$ is therefore

$$\left.\begin{aligned}&G''(\theta)+\lambda G(\theta)=0\\&G(\pi)=G(-\pi)\\&G'(\pi)=G'(-\pi)\end{aligned}\right\} \qquad ...(2)$$

This is a periodic Strum-Liouville problem and first we solve it by considering different cases:

**Case 1:** $\lambda=0$

In this case, the equation reduces to

$$G''(\theta)=0$$

with the general solution

$$G(\theta)=c+d\theta$$

Now,

$$G(\pi)=G(-\pi)\Rightarrow c+d\pi=c-d\pi\Rightarrow 2d\pi=0$$
$$\Rightarrow d=0$$
$$\therefore G(\theta)=c$$

which satisfies $G'(\pi)=G'(-\pi)$

Thus, $\lambda=0$ is an eigen value of the problem with eigen function

$$G(\theta)=c_0=\text{constant}$$

**Case 2:** $\lambda<0$

Let $\lambda=-n^2$

Then, the differential equation (2) is

$$G''(\theta) - n^2 G(\theta) = 0$$

with the general solution given by

$$G(\theta) = ce^{n\theta} + de^{-n\theta}$$

Now,

$$G(\pi) = G(-\pi) \Rightarrow ce^{n\pi} + de^{-n\pi} = ce^{-n\pi} + de^{n\pi}$$
$$\therefore G(\theta) = c\left(e^{n\theta} + e^{-n\theta}\right) \Rightarrow c - d = 0 \Rightarrow c = d$$

Also,

$$G'(\pi) = G'(-\pi) \Rightarrow cn(e^{n\pi} - e^{-n\pi}) = cn(e^{-n\pi} - e^{n\pi})$$
$$\Rightarrow 2cn = 0 \Rightarrow c = 0$$
$$\therefore G(\theta) = 0$$

Thus, we have no eigen value in this case.

**Case 3:** $\lambda > 0$

Let $\lambda = k^2$. Then, the differential equation (2) is

$$G''(\theta) + k^2 G(\theta) = 0$$

with the general solution given by

$$G(\theta) = c\cos(k\theta) + d\sin(k\theta)$$

Now,

$$G(\pi) = G(-\pi) \Rightarrow c\cos(k\pi) + d\sin(k\pi) = c\cos(k\pi) - d\sin(k\pi)$$
$$\Rightarrow 2d\sin(k\pi) = 0$$

For a non-trivial solution, we take

$$k\pi = n\pi \quad \text{for n=1,2...}$$
$$\Rightarrow k = n \quad \text{for n=1,2...}$$

Similarly, result holds for $G'(\pi) = G'(-\pi)$

Thus, the general solution is given by

$$G_n(\theta) = c_n\cos(n\theta) + d_n\sin(n\theta)$$

Thus, the eigen values for the SLBVP (2) is

$$\lambda = n^2 \text{ ; n=0,1,2,3...}$$

and the eigen function is

$$G_0(\theta) = c_0$$
$$G_n(\theta) = c_n\cos(n\theta) + d_n\sin(n\theta)$$

Now, let $\lambda = n^2$ to get (1) as

$$rF''(r) + rF'(r) - n^2F(r) = 0$$

This is a second order Euler differential equation with general solution

$$F_n(r) = a_n r^n + b_n r^{-n} \quad , \text{ for n=1,2,3...}$$
and $F_0(r) = a_0 = \text{constant}$ , for n=0

The requirement that the solution must be bounded forces to choose each $b_n = 0$ because $r^{-n} \to \infty$ as $r \to 0^+$ (centre of the disk).

Combining cases, we can write

$$F_n(r) = a_n r^n \text{ for n=0,1,2...}$$

For n=0,1,2…, we now have functions of the form

$$u_n(r,\theta) = F_n(r)G_n(\theta) = a_n r^n\left[c_n\cos(n\theta) + d_n\sin(n\theta)\right]$$

Setting $A_n = a_n c_n$ and $B_n = a_n d_n$, we have

$$u_n(r,\theta) = r^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$$

These functions satisfy Laplace's equation and the periodicity conditions, as well as the condition that solutions must be bounded. For any given n, this function will generally not satisfy the initial condition

$$u(R,\theta) = f(\theta)$$

For this, use the superposition

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$$

Now, the initial condition requires that

$$u(R,\theta) = f(\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n R^n\cos(n\theta) + B_n R^n\sin(n\theta)\right]$$

This is the Fourier series expansion of $f(\theta)$ on $[-\pi,\pi]$. Thus, choose

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

$$A_n R^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta \Rightarrow A_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta$$

and $\mathrm{B}_n R^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta \Rightarrow B_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta$

**Example:** As a specific example, suppose the disk has radius 3 and that $f(\theta)=2+\theta$. A routine integration gives

$$A_0=2\ ,\ A_n=0 \text{ for n=1,2,3...}$$

$$\text{and } B_n=\frac{2}{n.3^n}(-1)^{n+1}$$

The solution for this condition is

$$u(r,\theta)=2+\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\left(\frac{r}{3}\right)^n\sin(n\theta)$$

for $0\le r\le 3$ and $-\pi\le\theta\le\pi$.

**Problems:**

1. Find the steady-state temperature for a thin disk

i. of radius R with temperature on boundary is $f(\theta)=\cos^2\theta$ for $-\pi\le\theta\le\pi$

ii. of radius 1 with temperature on boundary is $f(\theta)=\cos^3\theta$ for $-\pi\le\theta\le\pi$

iii. of radius R with temperature on boundary is constant T.

2. Use the solution of steady-state temperature distribution in a thin disk to show that the temperature at the centre of disk is the average of the temperature values on the circumference of the disk.

[Hint: For temperature on the centre of disk, we let $r\to 0^+$, so that $u(r,\theta)=A_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}f(\theta)d\theta$

which is the average of $f(\theta)$, the temperature on the circumference of the disk.]

3. Find the steady-state temperature in the flat wedge-shaped plate occupying the region $0\le r\le k$ , $0\le\theta\le\alpha$ (in polar co-ordinates). The sides $\theta=0$ and $\theta=\alpha$ are kept at temperature zero and the ark $r=k$ for $0\le\theta\le\alpha$ is kept at temperature T.

[Hint: The BVP for this situation is

$$\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}=0$$

$$u(r,0)=u(r,\alpha)=0\ (0\le r\le k)$$

$$u(k,\theta)=T\ (0\le\theta\le\alpha)$$

### 1.3.3 Steady-State Temperature Distribution in a Semi-infinite Strip

Find the steady-state temperature distribution in a semi-infinite strip $x\ge 0,\ 0\le y\le 1,$ pictured in figure. The temperature on the top side and bottom side are kept at zero, while the left side is kept at temperature T.

The boundary value problem modelling this problem is:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (0 \le y \le 1, x \ge 0)$$
$$u(x,0)=0=u(x,1) \quad (\mathrm{x} \ge 0)$$
$$\mathrm{u(0,y)=T} \qquad (0 \le y \le 1)$$

Put $u(x,y)=X(x)Y(y)$ into Laplace's equation to get

$$X''Y+XY''=0 \;\Rightarrow\; \frac{X''}{X}=\frac{-Y''}{Y}$$

Since the left side depends only on x and right side only on x, and these variables are independent, both sides must equal the same constant:

$$\frac{X''}{X}=\frac{-Y''}{Y}=\lambda$$

Now, use the boundary conditions:

$$u(x,0)=X(x)Y(0)=0 \Rightarrow Y(0)=0$$
$$u(x,1)=X(x)Y(1)=0 \Rightarrow y(1)=0$$

Therefore, X must satisfy

$$X''-\lambda X=0$$

and, Y must satisfy

$$Y''+\lambda Y=0$$
$$Y(0)=Y(1)=0$$

The solution for the equation for $Y(y)$ is given by (by above article)

$$Y_n(y)=a_n\sin(n\pi y) \quad \text{for n=1,2,...}$$

with the eigen value given by

$$\lambda_n=n^2\pi^2$$

The problem for $X(x)$ is now

$$X''-n^2\pi^2X=0$$

The general solution of the differential equation is

$$X_n(x)=b_ne^{n\pi x}+c_ne^{-n\pi x}$$

Now, since $u(x,y)<\infty$, so $b_n=0$, otherwise $X_n(x)\to\infty$ as $\mathrm{x}\to\infty$. Thus, we have

$$X_n(x)=c_ne^{-n\pi x}$$

Thus, solution for each n is

$$u_n(x,y) = d_n e^{-n\pi x} \sin(n\pi y)\text{ , where } d_n = a_n c_n$$

For each n, using the superposition, we have

$$u(x,y) = \sum_{n=1}^{\infty} d_n e^{-n\pi x} \sin(n\pi y)$$

We want to choose the constant $d_n$, so that

$$u(0,y) = T = \sum_{n=1}^{\infty} d_n \sin(n\pi y)$$

which is Fourier sine expansion of T on [0,1]. Therefore, choose the entire co-efficient of $\sin(n\pi y)$ as the Fourier sine co-efficient:

$$\begin{aligned} d_n &= 2\int_0^1 T \sin(n\pi y)dy \qquad \text{[As in above article]} \\ &= \frac{2T}{n\pi}\left[1-(-1)^n\right] \end{aligned}$$

**Problem:**

1. Find a steady-state temperature distribution in the semi-infinite region $0 \le x \le a, y \ge 0$ if the temperature on the bottom and left sides are at zero and the temperature on the right side is kept at constant T.

2. Find the steady-state temperature distribution in the semi-infinite region $0 \le x \le 4, y \ge 0$ if the temperature on the vertical sides are kept at constant T and temperature on the bottom side is kept at zero.

   [Hint: Assume two semi-infinite regions, first with left end at temperature T and right end and bottom at temperature zero, second with right end at temperature zero and left end and bottom at temperature zero. Sum of these two solutions is the solution of the original problem.]

3. Use your intuition to guess the steady-state temperature in a thin rod of length L if the ends are perfectly insulated and the initial temperature is $f(x)$ for $0 < x < L$.

   [Hint: The boundary value problem modelling this problem is

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad (0 < x < L)\ (t > 0)$$

$$\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \qquad (t > 0)$$

$$u(x,0) = f(x) \qquad (0 < x < L)$$

### 1.3.4 Steady-State Temperature in a Semi-infinite Plate

The B.V.P. is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (0\le y\le b,\quad x\ge 0)$$
$$u(x,0)=0=u(x,b) \qquad (x\ge 0)$$
$$u(0,y)=T \qquad (0\le y\le b)$$

Put $u(x,y)=X(x)Y(y)$ into the given Laplace equation, we obtain

$$X''Y+XY''=0 \Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}$$

Since the left side is depend on $x$ only while right hand side is $y$ only. So both side must be equal to some constant. Let the constant of separation coefficient is $\lambda$. The above equation becomes

$X''-\lambda X=0$ and $Y''+\lambda Y=0$

And the boundary condition
$$u(x,0)=X(x)Y(0)=0 \Rightarrow Y(0)=0$$
$$u(x,b)=X(x)Y(b)=0 \Rightarrow Y(b)=0$$

Here we have more information for problem $Y$ with equations

$$Y''+\lambda=0$$
$$Y(0)=0 \quad and \quad Y(b)=0$$

In earlier article, we solve such problem and preceding like that, we have solution

$$Y_n=a_n\sin\left(\frac{n\pi y}{b}\right) \quad for \ n=1,2,3...$$

with the eigen value $\lambda_n=\dfrac{n^2\pi^2}{b^2}$.

Now the problem for X is

$$X''-\frac{n^2\pi^2}{b^2}X=0$$

The general solution is

$$X_n(x)=b_n e^{\frac{n\pi x}{b}}+c_n e^{-\frac{n\pi x}{b}}$$

For a bounded solution in the given domain, we have to assume $b_n=0$. Now the solution becomes $X_n(x)=c_n e^{-\frac{n\pi x}{b}}$. Thus the solution for each n by using the superposition is

$$u(x,y)=\sum_{n=1}^{\infty} d_n e^{-\frac{n\pi x}{b}} \sin\left(\frac{n\pi y}{b}\right) \text{ where } d_n = a_n c_n$$

Now using the condition $u(0,T)=T$, we have

$u(0,y)=T=\sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi y}{b}\right)$ which is a Fourier sine expansion of T on [0,1]. The coefficient

$$d_n = 2\int_0^b T\sin\left(\frac{n\pi y}{b}\right)dy$$

$$=\frac{2T}{n\pi}\left[1-(-1)^n\right]$$

$$u(x,y)=\sum_{n=1}^{\infty}=\frac{2T}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi y}{b}\right)e^{\frac{-n\pi x}{b}}$$

### 1.3.5 Steady-State Temperature in an Infinite Plate

Suppose we want the steady-state temperature distribution in a thin, flat plate extending over the right quarter plane $x \geq 0, y \geq 0$. Assume that the temperature on the vertical side $x=0$ is kept at zero, while the bottom side $y=0$ is kept at a temperature $f(x)$.

The BVP modelling this problem is:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad (x>0, y>0)$$
$$u(0,y)=0 \quad (y>0)$$
$$u(x,0)=f(x) \quad (x>0)$$

Now solving as in previous examples we get

$$u(x,y)=\int_0^{\infty} c_k \sin(kx)e^{-ky}dk$$

Finally, we require that

$$u(x,0)=f(x)=\int_0^{\infty} c_k \sin(kx)dk$$

This is the Fourier sine integral of $f(x)$ on $[0,\infty)$, so choose

$$c_k=\frac{2}{\pi}\int_0^{\infty} f(\xi)\sin(k\xi)d\xi$$

Thus, the solution for the problem is

$$u(x,y)=\int_0^\infty\left[\frac{2}{\pi}\int_0^\infty f(\xi)\sin(k\xi)d\xi\right]\sin(kx)e^{-ky}dk$$

**Example:** Assume that in the above problem

$$f(x)=\begin{cases}4\ , & 0\le x\le 2\\ 0\ , & x>2\end{cases}$$

Then,

$$\begin{aligned}c_k &=\frac{2}{\pi}\int_0^\infty f(\xi)\sin(k\xi)d\xi\\ &=\frac{2}{\pi}\int_0^\infty 4\sin(k\xi)d\xi\\ &=\frac{8}{\pi k}\left[1-\cos(2k)\right]\end{aligned}$$

Thus,

$$u(x,y)=\frac{8}{\pi}\int_0^\infty\left[\frac{1-\cos 2k}{k}\right]\sin(kx)e^{-ky}dk$$

## 1.4 Heat Equation in Unbounded Domains

Here, we will discuss the problems of temperature distribution in a bar with the space variable extending over the real line or half line.

### 1.4.1 Heat Conduction in a Semi-Infinite Bar

Suppose we want the temperature distribution in a Bar stretching from $0$ to $\infty$ along the x-axis. The left end is kept at temperature zero and the initial temperature in the cross-section at x is $f(x)$.

The boundary value problem for the temperature distribution is:

$$\begin{aligned}&\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \qquad (x>0,t>0)\\ &u(x,0)=f(x) \quad (x>0)\\ &u(0,t)=0 \qquad (t>0)\end{aligned}$$

As usual, we seek a solution, which is bounded.

Set,

$$\begin{aligned}&u(x,t)=X(x)T(t) \text{ to get}\\ &X''+\lambda X=0 \ \ (x>0)\\ &T'+\lambda a^2T=0 \ \ (t>0)\end{aligned}$$

Now as in previous examples, we get

$$u(x,t) = d_k \sin(kx) e^{-a^2k^2t}, \text{ where } d_k = a_k b_k$$

Now, using the superposition

$$u(x,t) = \int_0^\infty d_k \sin(kx) e^{-a^2k^2t} dk \qquad ...(1)$$

Finally, we must satisfy the initial condition:

$$f(x) = u(x,0) = \int_0^\infty d_k \sin(kx) dk \qquad ...(2)$$

For this choice the $d_k$'s are the Fourier sine integral co-efficient of $f(x)$; so

$$d_k = \frac{2}{\pi} \int_0^\infty f(\xi) \sin(k\xi) d\xi$$

With this choice of the co-efficient, the function defined by (1) is a solution of the problem.

**Example:** Suppose

$$f(x) = \begin{cases} \pi - x \ , & 0 \le \text{x} \le \pi \\ 0 \ , & x > \pi \end{cases}$$

Then, $d_k = \frac{2}{\pi} \int_0^\pi (\pi - \xi) \sin(k\xi) d\xi = \frac{2}{k}\left(1 - \frac{\sin(k\pi)}{k\pi}\right)$

The solution is

$$u(x,t) = \int_0^\infty \frac{2}{k}\left(1 - \frac{\sin(k\pi)}{k\pi}\right) \sin(kx) e^{-k^2\pi^2t} dk$$

### 1.4.2 Heat Conduction in Infinite Bar

Suppose we want the temperature distribution in a Bar stretching from $-\infty$ to $\infty$ along the x-axis. The initial temperature in the cross-section at x is $f(x)$. The boundary value problem for the temperature distribution is:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty, t > 0)$$
$$u(x,0) = f(x) \quad (-\infty < x < \infty)$$

There are no boundary conditions, so we impose the physically realistic condition that solutions should be bounded. As usual, we seek a solution, which is bounded.

Set,

$u(x,t) = X(x)T(t)$ to get

$X'' + \lambda X = 0 \ \ (-\infty < x < \infty)$

$T' + \lambda a^2 T = 0 \ (t > 0)$

Now as in previous examples, we get

$$u_k(x,t) = (a_k \cos(kx) + b_k \sin(kx))e^{-a^2k^2t},$$

that satisfy the Heat equation and are bounded on the real line over all k>0. Now, using the superposition

$$u(x,t) = \int_0^\infty (a_k \cos(kx) + b_k \sin(kx))e^{-a^2k^2t}dk \qquad ...(1)$$

Finally, we must satisfy the initial condition:

$$f(x) = u(x,0) = \int_0^\infty (a_k \cos(kx) + b_k \sin(kx))dk \qquad ...(2)$$

For this choice the $a_k$'s and $b_k$'s are the Fourier sine integral co-efficient of $f(x)$; so

$$a_k = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(k\xi)d\xi$$

and

$$a_k = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(k\xi)d\xi$$

With this choice of the co-efficient, the function defined by (1) is a solution of the problem.

## 1.5 Solution of Heat, Laplace and Wave Equations

### 1.5.1 Solution of Three-Dimensional Heat Equations in Cartesian co-ordinates

It is a partial differential equation of the form:

$$\frac{\partial u}{\partial t} = h^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \qquad ...(1)$$

To find its solution by the method of separation of variables, suppose that the solution of (1) is

$$u(x,y,z,t) = X(x)Y(y)Z(z)T(t) \qquad ...(2)$$

where $X(x)$ is a function of x only, $Y(y)$ is a function of y only, $Z(z)$ is a function of z only and $T(t)$ is a function of t only.

We get on separating the variables

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Y}\frac{d^2Z}{dz^2} = \frac{1}{h^2T}\frac{dT}{dt} \qquad ...(3)$$

Choosing the constant of separation such that

$$\frac{1}{X}\frac{d^2X}{dx^2} = -p_1^2, \frac{1}{Y}\frac{d^2Y}{dy^2} = -p_2^2, \frac{1}{Z}\frac{d^2Z}{dz^2} = -p_3^2 \text{ and } \frac{1}{h^2T}\frac{dT}{dt} = -p^2, \text{where } p^2 = p_1^2 + p_2^2 + p_3^2$$

Thus, we have the following three equations

$$\frac{d^2X}{dx^2} + p_1^2 X = 0$$

$$\frac{d^2Y}{dy^2} + p_2^2 Y = 0$$

$$\frac{d^2Z}{dz^2} + p_3^2 Z = 0$$

$$\frac{dT}{dt} + p^2 h^2 T = 0$$

with the solutions

$$X(x) = A\cos p_1 x + B\sin p_1 x$$

$$Y(y) = C\cos p_2 y + D\sin p_2 y$$

$$Y(y) = E\cos p_3 z + F\sin p_3 z$$

$$T(t) = Ge^{-p^2h^2t} = Ge^{-(p_1^2+p_2^2+p_3^2)h^2t}$$

Combining these solutions and using the superposition, we get

$$u(x,y,z,t) = \sum_{p_1,p_2,p_3=1}^{\infty} (A\cos p_1x + B\sin p_1x)(C\cos p_2y + D\sin p_2y)(E\cos p_3z + F\sin p_3z)Ge^{-(p_1^2+p_2^2+p_3^2)h^2t}$$

**Corollary:** The Heat equation in two-dimensional is

$$\frac{\partial u}{\partial t} = h^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

The solution is

$$u(x,y,t) = \sum_{p_1,p_2=1}^{\infty} (A\cos p_1x + B\sin p_1x)(C\cos p_2y + D\sin p_2y)Ee^{-(p_1^2+p_2^2)h^2t}$$

### 1.5.2 Solution of Heat Equation in Cylindrical Polar Co-ordinates

In cylindrical co-ordinates, Heat equation has the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{h^2}\frac{\partial u}{\partial t} \qquad ...(1)$$

To solve it by the method by separation of variables, we have

$$u(r,\theta,z,t) = R(r)\Theta(\theta)Z(z)T(t) \qquad ...(2)$$

giving

$$\frac{\partial u}{\partial r}=\frac{dR}{dr}\Theta(\theta)Z(z)T(t) \qquad , \qquad \frac{\partial^2 u}{\partial r^2}=\frac{d^2R}{dr^2}\Theta(\theta)Z(z)T(t)$$

$$\frac{\partial^2 u}{\partial \theta^2}=R(r)\frac{d^2\Theta}{d\theta^2}Z(z)T(t) \qquad , \qquad \frac{\partial^2 u}{\partial z^2}=R(r)\Theta(\theta)\frac{d^2Z}{dz^2}T(t)$$

$$\frac{\partial u}{\partial t}=R(r)\Theta(\theta)Z(z)\frac{dT}{dt}$$

Substituting all these values in equation (1), we get

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{d^2Z}{dz^2}=\frac{1}{h^2T}\frac{dT}{dt}$$

Using the method of separation of variables, we have

$$\frac{1}{h^2T}\frac{dT}{dt}=-\lambda^2 \Rightarrow \frac{dT}{dt}+\lambda^2h^2t=0 \qquad (3)$$

$$\frac{d^2Z}{dz^2}=-\kappa^2 \Rightarrow \frac{d^2Z}{dz^2}+\kappa^2h^2t=0 \qquad (4)$$

and $$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=-\mu^2 \Rightarrow \frac{d^2\Theta}{d\theta^2}+\mu^2\Theta=0 \qquad ...(5)$$

so that

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)-\frac{\mu^2}{r^2}-\kappa^2=-\lambda^2$$

$$\Rightarrow \frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\varepsilon^2-\frac{\mu^2}{r^2}\right)R=0 \qquad \text{where } \varepsilon^2=\lambda^2\text{-}\kappa^2 \qquad ...(6)$$

with solution of (3) as

$$T(t)=ae^{-h^2\lambda^2t}$$

solution of (4) as

$$Z(z)=b\cos(\kappa z)+c\sin(\kappa z)$$

solution of (5) as

$$\Theta(\theta)=e\cos(\mu\theta)+f\sin(\mu\theta)$$

The equation (6) is Modified Bessel's Equation and the solution is

$$R(r)=AJ_{\mu}(\varepsilon r)+BJ_{-\mu}(\varepsilon r) \qquad \text{for fractional } \mu$$

and

$$R(r)=AJ_{\mu}(\varepsilon r)+BY_{-\mu}(\varepsilon r) \qquad \text{for integral } \mu$$

where

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{(n+1)_r} \quad , \quad \text{where } (n+1)_r = (n+1)(n+2)\ldots(n+r)$$

$$J_{-n}(x) = (-1)^n J_n(-x)$$

and

$$Y_n(x) = \frac{2}{\pi}\left\{\log\frac{x}{2} + \gamma\right\} J_n(x) - \frac{1}{\pi}\sum_{p=0}^{n-1} \frac{\overline{|n-p}}{\underline{|p}} \left(\frac{2}{x}\right)^{n-2p}$$

Thus, the solution of Heat equation is

$$u(r,\theta,z,t) = \sum_{\lambda,\kappa,\mu} ae^{-h^2\lambda^2 t}\left[b\cos(\mu\theta) + c\sin(\mu\theta)\right]\left[e\cos(\kappa z) + f\sin(\kappa z)\right]\left[AJ_\mu(\varepsilon r) + BJ_{-\mu}(\varepsilon r)\right]$$

for fractional $\mu$,

and $u(r,\theta,a,t) = \sum_{\lambda,\kappa,\mu} ae^{-h^2\lambda^2 t}\left[b\cos(\mu\theta) + c\sin(\mu\theta)\right]\left[e\cos(\kappa z) + f\sin(\kappa z)\right]\left[AJ_\mu(\varepsilon r) + BY_\mu(\varepsilon r)\right]$

for integral $\mu$.

**Corollary:** In 2-dimesnion, the cylindrical heat Equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{h^2}\frac{\partial u}{\partial t}$$

and the solution of Heat equation is

$u(r,\theta,t) = \sum_{\lambda,\mu} ae^{-h^2\lambda^2 t}\left[b\cos(\mu\theta) + c\sin(\mu\theta)\right]\left[AJ_\mu(\lambda r) + BJ_{-\mu}(\lambda r)\right]$ for fractional $\mu$,

and $u(r,\theta,t) = \sum_{\lambda,\mu} ae^{-h^2\lambda^2 t}\left[b\cos(\mu\theta) + c\sin(\mu\theta)\right]\left[AJ_\mu(\lambda r) + BY_\mu(\lambda r)\right]$ for integral $\mu$.

### 1.5.3 Solution of Heat Equation in Spherical Co-ordinates

In spherical polar co-ordinates, it has the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{h^2}\frac{\partial u}{\partial t} \qquad \ldots(1)$$

Assuming $u(r,\theta,\phi,t) = R(r)\Theta(\theta)\Phi(\phi)T(t)$, equation (1) becomes

$$\left[\frac{1}{R}\frac{d^2R}{dr^2} + \frac{2}{Rr}\frac{dR}{dr} + \frac{1}{\Theta r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{r^2\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}\right] = \frac{1}{h^2}\frac{dT}{dt}$$

Let $\qquad \frac{1}{h^2 T}\frac{dT}{dT} = -\lambda^2 \;\Rightarrow\; \frac{dT}{dt} + \lambda^2 h^2 T = 0 \qquad \ldots(2)$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2 \quad \Rightarrow \frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \qquad \ldots(3)$$

with solution given by

$$T(t) = ae^{-\lambda^2 h^2 t}$$
$$\Theta(\phi) = be^{\pm im\phi}$$

and

$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\left(\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr}\right) - \lambda^2 r^2 = n(n+1) \text{ (say)}$$

giving

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \left(\lambda^2 r^2 - n(n+1)\right)R = 0 \qquad \text{...(4)}$$
$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \qquad \text{...(5)}$$

Here (4), being homogeneous, if we put $r = e^s$ and $D \equiv \frac{d}{ds}$, reduces to

$$\left[D(D-1) + 2D - n(n+1)\right]R = 0$$
or $\quad (D-n)(D+(n+1))R = 0$
$$\therefore \quad R = Ae^{ns} + Be^{-(n+1)s}$$
$$= Ar^n + Br^{-n-1}$$

Putting $\mu = \cos\theta$ in (5), so that

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta\frac{d\Theta}{d\mu} \Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta} = -\frac{d}{d\mu}$$

we have

$$\frac{d}{d\mu}\left\{\left(1-\mu^2\right)\frac{d\Theta}{d\mu}\right\} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0$$

or $$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0$$

which is associated Legendre equation, then the solution is of the form

$$\Theta = \Theta(\cos\theta)$$

and hence solution of given problem is

$$u_n(r,\theta,\phi,t) = (Ar^n + Br^{-n-1})\Theta(\cos\theta)e^{\pm im\phi}e^{-\lambda^2 h^2 t}$$

Hence, summing overall n and trying superposition, the general solution of (1) may be expressed as

$$u(r,\theta,\phi,t) = \sum_{n,\lambda,m} (A_n r^n + \frac{B_n}{r^{n+1}})\Theta(\cos\theta)e^{\pm i\lambda\phi}e^{-\lambda^2 h^2 t}$$ is required solution.

### 1.5.4 Solution of Laplace Equation in Cartesian Co-ordinates

In Cartesian co-ordinates, the Laplace equation has the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad ...(1)$$

To solve it by the method of separation of variables, we have

$$V(x, y, z) = X(x)Y(y)Z(z) \qquad ...(1)$$

giving $\frac{\partial^2 V}{\partial x^2} = \frac{d^2 X}{dx^2}$YZ, $\quad \frac{\partial^2 V}{\partial y^2} = X\frac{d^2 Y}{dy^2}$ Z and $\quad \frac{\partial^2 V}{\partial z^2} = XY\frac{d^2 Z}{dz^2}$

so that (1) gives

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -p_1^2 \Rightarrow \frac{d^2 X}{dx^2} + p_1^2 X = 0 \qquad ...(2)$$

$$\frac{1}{Y}\frac{d^2 Y}{dy^2} = -p_2^2 \Rightarrow \frac{d^2 Y}{dy^2} + p_2^2 Y = 0 \qquad ...(3)$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = p^2 \Rightarrow \frac{d^2 Z}{dz^2} - p^2 Z = 0 \qquad \text{where } p^2 = p_1^2 + p_2^2 \qquad ...(4)$$

The solutions of these equations are

$$X(x) = A\cos p_1 x + B\sin p_1 x$$
$$Y(y) = C\cos p_2 y + D\sin p_2 y$$
$$Z(z) = Ee^{pz} + Fe^{-pz}$$

The combined solution of (1) is

$$V_p(x, y, z) = (A\cos p_1 x + B\sin p_1 x)(C\cos p_2 y + D\sin p_2 y(ce^{pz} + De^{-pz})$$

Using the superposition, we have

$$V(x, y, z) = \sum_{p_1, p_2} (A\cos p_1 x + B\sin p_1 x)(C\cos p_2 y + D\sin p_2 y(ce^{pz} + De^{-pz})$$

**Corollary**: In 2-dimesnion, the Laplace equation has the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad ...(1)$$

To solve it by the method of separation of variables, we have

$$V(x, y) = X(x)Y(y) \qquad ...(2)$$

giving $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{d^2 X}{dx^2}\text{Y}$ and $\dfrac{\partial^2 V}{\partial y^2} = X\,\dfrac{d^2 Y}{dy^2}$

so that (1) gives

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2} = -p^2$$

Now,

$$\frac{d^2 X}{dx^2} + p^2 X = 0 \qquad ...(3)$$

and $$\frac{d^2 Y}{dy^2} - p^2 Y = 0 \qquad ...(4)$$

The solutions of these equations are

$$X(x) = A\cos px + B\sin px$$
$$Y(y) = Ce^{py} + De^{-py}$$

The combined solution of (1) is

$$V_p(x, y) = (A\cos px + B\sin px)(ce^{py} + De^{-py})$$

Using the superposition, we have

$$V(x, y) = \sum_p [(A\cos px + B\sin px)(ce^{py} + De^{-py})]$$

**1.5.5 Solution of Three-Dimensional Laplace Equation in Cylindrical Co-ordinates**

In cylindrical co-ordinates, Laplace's equation has the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad ...(1)$$

Assuming that $V(r, \theta, z) = R(r)\Theta(\theta)Z(z)$, then (1) yields

$$\frac{1}{R}\frac{d^2 R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0 \qquad ...(2)$$

Since the variables are separated, we can take

$$\frac{1}{z}\frac{\partial^2 Z}{\partial z^2} = \lambda^2 \quad \text{and} \quad \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -\mu^2$$

$$\Rightarrow \frac{\partial^2 Z}{\partial z^2} - \lambda^2 z = 0 \quad \text{and} \quad \frac{d^2\Theta}{d\theta^2} + \mu^2\Theta = 0$$

yielding the general solutions as

$$Z(z) = Ae^{\lambda z} + Be^{-\lambda z} \quad \text{and} \quad \Theta(\theta) = C\cos\mu\theta + D\sin\mu\theta$$

Now, equation (2) reduces to

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{\mu^2}{r^2}\right)R = 0$$

Which is Bessel's modified equation, having the solution

$R(r) = EJ_\mu(\lambda r) + FJ_{-\mu}(\lambda r)$ for fractional $\mu$

and $R(r) = EJ_\mu(\lambda r) + FY_\mu(\lambda r)$ for integral $\mu$.

Hence, the combined solution is

$$V(r,\theta,z) = \sum_{\lambda,\mu}\left(Ae^{\lambda z} + Be^{-\lambda z}\right)\left(C\cos\mu\theta + D\sin\mu\theta\right)\left(EJ_\mu(\lambda r) + FJ_{-\mu}(\lambda r)\right),$$

for fractional $\mu$.

$$V(r,\theta,z) = \sum_{\lambda,\mu}\left(Ae^{\lambda z} + Be^{-\lambda z}\right)\left(C\cos\mu\theta + D\sin\mu\theta\right)\left(EJ_\mu(\lambda r) + FY_\mu(\lambda r)\right),$$

for integral $\mu$.

**Corollary: 1.** Taking constant $A_{\lambda\mu}$ and $B_{\lambda\mu}$, the general solution can be written as

$$R(r) = A_{\lambda\mu}J_\mu(\lambda r) + B_{\lambda\mu}Y_\mu(\lambda r)$$

But $Y_\mu(\lambda r) \to \infty$ as $r \to 0$, therefore if it is finite along the line $r = 0$, then $B_{\lambda\mu} = 0$, hence the solution is

$$V(r,\theta,z) = \sum_\lambda\sum_\mu A_{\lambda\mu}J_\mu(\lambda r)e^{\pm\lambda z \pm i\mu\theta}$$

Trying the superposition, we can write the solution as:

$$V(r,\theta,z) = \sum_{\lambda,\mu=0}^{\infty} J_\mu(\lambda r)\left[e^{\lambda z}(A_\mu\cos\mu\theta + B_\mu\sin\mu\theta) + e^{-\lambda z}(C_\mu\cos\mu\theta + D_\mu\sin\mu\theta\right]$$

2. **Solution of Laplace Equation in Two Dimension in Polar Co-ordinates**

The Laplace equation has the form:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} = 0 \qquad ...(1)$$

To solve it by the method of separation of variables, we take

$$V\left(r,\theta\right) = R(r)\Theta(\theta) \qquad ...(2)$$

giving

$$\frac{\partial V}{\partial r}=\frac{dR}{dr}\,\Theta(\theta) \quad ; \quad \frac{\partial^2 V}{\partial r^2}=\frac{d^2R}{dr^2}\Theta(\theta) \quad \text{and}$$

$$\frac{\partial^2 V}{\partial \theta^2}=R(r)\frac{d^2\Theta}{d\theta^2}$$

Substituting all these in the equation (1), we get

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}\right)=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=n^2(\text{say})$$

so that we have

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}\right)=n^2(\text{say})$$

$$\Rightarrow\ r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-n^2R=0 \qquad ...(3)$$

which is homogeneous and hence on putting

$$r=e^z, \quad \text{so that} \quad z=\log r \text{ and } \quad D=r\frac{d}{dr}=\frac{d}{dz}$$

then the equation (3) reduces to

$$\left[D(D-1)+D-n^2\right]r=0$$

$$\Rightarrow\left(D^2-n^2\right)r=0$$

Its auxiliary equation is

$$D^2-n^2=0$$

$$\Rightarrow D=\pm\, n$$

$$\therefore R(r)=Ae^{nz}+Be^{-nz}$$

$$=Ar^n+Br^{-n}$$

Also, the equation for (1) is

$$-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=n^2 \qquad ...(4)$$

It has the solution

$$\Theta(\theta)=C\cos n\theta+D\sin n\theta$$

The combined solution is

$$V_n\left(r,\theta\right)=\left(Ar^n+Br^{-n}\right)\left(C\cos n\theta+D\sin n\theta\right) \qquad ...(5)$$

Also, for n=0, (3) and (4) becomes

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} = 0 \quad \text{...(6)}$$

and $$\frac{d^2\Theta}{d\theta^2} = 0 \quad \text{...(7)}$$

Having the solution of (6) and (7) as

$$R(r) = c_1 + c_2 \log r$$
$$\Theta(\theta) = d_1 + d_2\theta$$

Thus, for n=0, the solution is

$$V(r,\theta) = [c_1 + c_2 \log r][d_1 + d_2\theta]$$

Thus, the general solution is

$$V(r,\theta) = [c_1 + c_2 \log r][d_1 + d_2\theta] + \sum_{n=1}^{\infty}\left[A_n r^n + B_n r^{-n}\right][C_n \cos n\theta + D_n \sin n\theta]$$

**1.5.5 Solution of Laplace Equation in Spherical Co-ordinates**

In spherical polar co-ordinates, it has the form

$$r^2 \frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \quad \text{...(1)}$$

Assuming $V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$, equation (1) becomes

$$\left[\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \lambda^2$$

$$\Rightarrow -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \lambda^2 \quad \Rightarrow \quad \frac{d^2\Phi}{d\phi^2} + \lambda^2\Phi = 0$$

with solution given by

$$\Theta(\phi) = Ce^{\pm i\lambda\phi}$$

and

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\lambda^2}{\sin^2\theta} = n(n+1) \text{ (say)}$$

giving

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 \quad \text{...(2)}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{\lambda^2}{\sin^2\theta}\right]\Theta = 0 \quad \text{...(3)}$$

Here (2), being homogeneous, if we put $r = e^s$ and $D \equiv \dfrac{d}{ds}$, reduces to

$$\left[D(D-1)+2D-n(n+1)\right]R=0$$

or $(D-n)(D+(n+1))R=0$

$$\therefore \quad R = Ae^{ns}+Be^{-(n+1)s}$$
$$= Ar^{n}+Br^{-n-1}$$

Putting $\mu = \cos\theta$ in (4), so that

$$\frac{d\Theta}{d\theta}=\frac{d\Theta}{d\mu}\frac{d\mu}{d\theta}=-\sin\theta\frac{d\Theta}{d\mu}\Rightarrow\frac{1}{\sin\theta}\frac{d}{d\theta}=-\frac{d}{d\mu}$$

we have

$$\frac{d}{d\mu}\left\{\left(1-\mu^2\right)\frac{d\Theta}{d\mu}\right\}+\left\{n(n+1)-\frac{\lambda^2}{1-\mu^2}\right\}\Theta=0$$

or $$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left\{n(n+1)-\frac{\lambda^2}{1-\mu^2}\right\}\Theta=0$$

which is associated Legendre equation, then the solution is of the form

$$\Theta=\Theta(\cos\theta)$$

and hence solution of given problem is

$$V_n(r,\theta,\phi)=(Ar^n+Br^{-n-1})\Theta(\cos\theta)e^{\pm i\lambda\phi}$$

Hence, summing overall n and trying superposition, the general solution of (1) may be expressed as

$$V(r,\theta,\phi)=\sum_{n,\lambda}(A_n r^n+\frac{B_n}{r^{n+1}})\Theta(\cos\theta)e^{\pm i\lambda\phi}$$ is required solution.

### 1.5.7 Solution of Three-Dimensional Wave Equation in Cartesian Co-ordinates

A partial differential equation of the form

$$\frac{\partial^2 u}{\partial t^2}=c^2\nabla^2 u$$

is known as Wave equation, that is

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\Rightarrow\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...(1)}$$

with the conditions

$$\frac{\partial u}{\partial x}=0 \text{ at } x=0, x=a$$

$$\frac{\partial u}{\partial y}=0 \text{ at } \mathrm{y}=0, y=a$$

$$\frac{\partial u}{\partial z}=0 \text{ at } \mathrm{z}=0, z=a$$

and $u(x,y,z,t)\neq 0$ at $t=0$

To solve the problem, we shall use the method of separation of variables and assume that

$$u(x,y,z,t)=X(x)Y(y)Z(z)T(t)$$

Now proceed as in previous examples to get

$$u_{n_1n_2n_3}(x,y,z,t)=\alpha_{n_1n_2n_3}\cos\frac{n_1\pi}{a}\cos\frac{n_2\pi}{a}\cos\frac{n_3\pi}{a}\cos\left(\frac{\pi ct}{a}\sqrt{n_1^2+n_2^2+n_3^2}\right)$$

Therefore, using the superposition, the general solution is

$$u(x,y,z,t)=\sum_{n_1,n_2,n_3=1}^{\infty}\alpha_{n_1n_2n_3}\cos\frac{n_1\pi}{a}x\cos\frac{n_2\pi}{a}y\cos\frac{n_3\pi}{a}z\cos\left(\frac{\pi ct}{a}\sqrt{n_1^2+n_2^2+n_3^2}\right)$$

**Corollary:** Wave equation in two-dimensional is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

And the solution is given by

$$u(x,y,t)=\sum_{n_1,n_2=1}^{\infty}\alpha_{n_1n_2}\cos\frac{n_1\pi}{a}x\cos\frac{n_2\pi}{a}y\cos\left(\frac{\pi ct}{a}\sqrt{n_1^2+n_2^2}\right)$$

**1.5.8 Solution of three-dimensional Wave equation in cylindrical co-ordinates**

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

Let the solution is $u(r,\theta,z,t)=R(x)\Theta(\theta)Z(z)T(t)$ ...(2)

Choosing the constant the separation of variable such that

$$\frac{1}{c^2T}\frac{d^2T}{dt^2}=-\mathrm{p}^2 \quad \Rightarrow \frac{d^2T}{dt^2}+\mathrm{p}^2c^2T=0 \qquad ...(3)$$

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=-q^2 \quad \Rightarrow \frac{d^2\Theta}{d\theta^2}+q^2\Theta=0 \qquad ...(4)$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=-s^2 \quad \Rightarrow \frac{d^2Z}{dz^2}+s^2Z=0 \qquad ...(5)$$

The equation (1) becomes

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{q^2}{r^2}-s^2=-p^2$$

$$\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}\right)+\left(\xi^2-\frac{q^2}{r^2}\right)R=0 \quad ...(6)$$

where $\xi^2=-s^2+p^2$. Equation (6) is the modified Bessel's equation of order $q$ has a solution

$$R(r)=A_3J_q(\xi r)+B_3J_{-q}(\xi r) \text{ for fractional } q$$

and $R(r)=A_3J_q(\xi r)+B_3Y_q(\xi r)$ for integral $q$.

Now

For a bounded solution, $Y_q(\xi r)\to\infty$ as $r\to 0$, therefore if it is finite along the line $r=0$, then $B_3=0$. Thus, the general solution of equation(1) is

$$u(r,\theta,z,t)=\sum_{p,q,s}CJ_q(\xi r)\left[A_1\cos(pct)+B_1\sin(pct)\right]\left[A_2\cos(q\theta)+B_2\sin(q\theta)\right]\left[A_3\cos(sz)+B_3\sin(sz)\right]$$

### 1.5.9 Solution of Three-dimensional Wave equation in Spherical co-ordinates

In polar spherical co-ordinates the Wave equation is

$$\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}=\frac{1}{c^2}\frac{\partial^2u}{\partial t^2}$$

Assuming that the solution of (1) is

$$u(r,\theta,\phi,t)=R(r)\Theta(\theta)\Phi(\phi)T(t)$$

Now proceed as in previous articles to get

$$u(r,\theta,\phi,t)=\sum_{p,q,s}\left(A_1e^{\pm im\phi}\right)\left(A_2e^{\pm ipct}\right)\left(CP_n^m(\cos\theta)+DP_n^{-m}(\cos\theta)\right)\left(Er^{-\frac{1}{2}}J_{n+\frac{1}{2}}(pr)+Fr^{-\frac{1}{2}}J_{-\left(n+\frac{1}{2}\right)}(pr)\right) \quad \textbf{1.6}$$

**Method of separation of variables to solve B.V.P. associated with motion of a vibrating string**

### 1.6.1 Solution of the problem of vibrating string with zero initial velocity and with initial displacement

Let us consider an elastic string of length $L$, fastened at its ends on the x-axis and assume that it vibrates in the $xy-$plane. Initially the string is released from the rest and we want to find out the expression for displacement function $y(x,t)$. The B.V.P. modeling the motion of string is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L,\ t > 0) \qquad ...(1)$$
$$y(0,t) = y(L,t) = 0 \ \ (t > 0) \qquad ...(2)$$
$$y(x,0) = f(x), \qquad (0 < x < L) \qquad ...(3)$$
$$\frac{\partial y}{\partial t}(x,0) = 0, \qquad (0 < x < L) \qquad ...(4)$$

Here, it is assumed that $f(x)$ is the initial displacement of the string before release and initial velocity is zero. We find the solution of equation (1) by separation of variables. For this, we set $y(x,t) = X(x)T(t)$ and using this, we get

$$XT'' = a^2 X''T \quad or \quad \frac{X''}{X} = \frac{T''}{a^2 T}$$

Since the left side of this equation is a function of x only and right hand side is a function of $t$ only where $x$ and $t$ are independent, both must equal to some constant. Let the constant of separation is $-\lambda$. The above equation has become

$$X'' + \lambda X = 0; \qquad T'' + \lambda a^2 T = 0 \quad ...(5)$$

Since $y(0,t) = X(0)T(t) = 0$

From here we conclude that $X(0) = 0$. This assumes that $T(t)$ is non-zero for some $t$. Otherwise $T = 0$ is zero for all time and we get the trivial solution, i.e., string would not move and it is possible only when $f(x) = 0$ means string is not displaced.

Similarly $y(L,t) = X(L)T(t) = 0$. Implies that $X(L) = 0$.

The problem for $X$ is

$$X'' + \lambda X = 0$$
$$X(0) = 0 = X(L)$$

We have solved such types of problems earlier and the solution is

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right) \text{ with eigen values } \lambda = \frac{n^2\pi^2}{L^2} \text{ for } n = 1, 2, 3...,$$

Now the problem of $T(t)$ is

$$T'' + \frac{n^2\pi^2}{L^2} T = 0$$

With the condition $\frac{\partial y(x,0)}{\partial t} = 0 \Rightarrow X(x)T'(0) = 0 \Rightarrow T'(0) = 0.$ Otherwise the solution becomes trivial.

Thus the general solution is

$$T_n(t) = C_n \cos\left(\frac{n\pi at}{L}\right) + D_n \sin\left(\frac{n\pi at}{L}\right)$$

By applying the condition $T'(0) = 0$, we get $D_n = 0$. Hence for fixed $n$, the solution for

$T_n(t) = C_n \cos\left(\frac{n\pi at}{L}\right)$ for $n = 1, 2, 3...$

Now, for a fixed $n$, the solution of equation (1) is

$$y_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right) \quad where\ B_n = A_n C_n$$

Using the superposition, we obtain

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right)$$

Now, using the condition (3), we have

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Which is Fourier sine series and the value of constant coefficient $B_n$ is $\frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$

Thus, we have

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right)$$

**Corollary:** In the above problem, if $f(x)$ is replaced by

$$f(x) = \begin{cases} x & , 0 \le x \le \frac{L}{2} \\ L - x & , \frac{L}{2} \le x \le L \end{cases}$$

The coefficient

$$\begin{aligned} B_n &= \frac{2}{L}\left[\int_0^{\frac{L}{2}} \xi \sin\left(\frac{n\pi\xi}{L}\right)d\xi + \int_{\frac{L}{2}}^{L} (L-\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right] \\ &= \frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Thus the solution becomes

$$y(x,t)=\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right)$$

Since $\sin\left(\frac{n\pi}{2}\right)=0$ if n is even and $\sin\left(\left(2k-1\right)\frac{\pi}{2}\right)=\left(-1\right)^{k-1}$ if n is odd positive integer. The solution is

$$y(x,t)=\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(-1\right)^{k-1}}{\left(2n-1\right)^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right)$$

### 1.6.2 Solution of the Problem of Vibrating String with Initial Velocity and Zero Initial Displacement

**The B.V.P is**

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}\qquad \left(0<x<L,\ t>0\right)\qquad ...(1)$$

$$y\left(0,t\right)=y\left(L,t\right)=0\qquad \left(t>0\right)\qquad ...(2)$$

$$y\left(x,0\right)=0,\qquad (0<x<L)\qquad ...(3)$$

$$\frac{\partial y}{\partial t}(x,0)=g(x),\qquad \left(0<x<L\right)\qquad ...(4)$$

Similarly to earlier article, the solution for $X$ is

$$X_n(x)=A_n\sin\left(\frac{n\pi x}{L}\right)\text{ with eigen values }\lambda=\frac{n^2\pi^2}{L^2}\text{ for }n=1,2,3...,$$

The solution for $T$ is

$$T_n(t)=C_n\cos\left(\frac{n\pi at}{L}\right)+D_n\sin\left(\frac{n\pi at}{L}\right)$$

And applying the condition (3), we have

$$y(x,0)=X(x)T(0)=0\Rightarrow T(0)=0$$

This implies $C_n=0$ and the solution for T is $T_n(t)=\sum_{n=1}^{\infty}D_n\sin\left(\frac{n\pi at}{L}\right)$

Therefore, the general solution is

$$y(x,t)=\sum_{n=1}^{\infty}y_n(x,t)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi at}{L}\right)\quad where\ A_nD_n=B_n\ \dots\ (5)$$

Now, using condition (4), we have

$$\frac{\partial y}{\partial t}(x,0) = g(x)$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Which is a Fourier sine series for $g(x)$, the value of coefficient $B_n$ is

$$B_n = \frac{2}{n\pi a}\left[\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]$$

$$\therefore \quad y(x,t) = \frac{2}{n\pi a}\sum_{n=1}^{\infty}\left[\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi at}{L}\right)$$

**Example**: Solve the following B.V.P

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L,\ t > 0) \qquad ...(1)$$

$$y(0,t) = y(L,t) = 0 \ \ (t > 0) \qquad ...(2)$$

$$y(x,0) = 0, \qquad (0 < x < L) \qquad ...(3)$$

$$\frac{\partial y}{\partial t}(x,0) = \begin{cases} x & ,0 \le x \le \dfrac{L}{4} \\ 0 & ,\dfrac{L}{4} \le x \le L \end{cases} \qquad ...(4)$$

### 1.6.3 The solution of the String Problem with Initial Velocity and with Displacement

Consider a string with both initial displacement $f(x)$ and initial velocity $g(x)$. To solve this problem, we firstly, formulate two separate problems, the first with initial displacement $f(x)$ and zero initial velocity, and the second with zero initial displacement and initial velocity $g(x)$. In earlier article, we solved the problem of string with zero initial velocity and with displacement and initial velocity and with zero displacement. Let $y_1(x,t)$ be the solution of the first problem, and $y_2(x,t)$ the solution of the second. Now let $y(x,t) = y_1(x,t) + y_2(x,t)$. Then $y$ satisfies the Wave equation and the boundary conditions.

## 1.7 Solution of Wave equation for Semi-infinite and Infinite Strings

### 1.7.1 Wave Motion for a Semi-infinite String

Let us consider an elastic string which is fixed at $x = 0$ and stretched from 0 to $\infty$. The B.V.P. for the motion of semi-infinite string is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (x > 0,\ t > 0) \qquad ...(1)$$

$$y(0,t) = 0 \qquad (t > 0) \qquad ...(2)$$

$$y(x,0) = f(x), \qquad (x > 0) \qquad ...(3)$$

$$\frac{\partial y}{\partial t}(x,0) = g(x), \qquad (x > 0) \qquad ...(4)$$

Here, this the problem of vibrating string with initial velocity and displacement. So we will separate the problem in two parts: (i) zero initial velocity and with displacement (ii) with initial velocity and zero displacement.

**(i) Zero initial velocity**

For this case $g(x) = 0$. For a bounded solution, we firstly set $y(x,t) = X(x)T(t)$.

Using this in equation, we get

$$\frac{X''}{X} = \frac{T''}{a^2 T} \qquad \ldots (5)$$

In equation (5), the left side is a function of $x$ only while right side is function of $t$. So each side must be equal to some constant, let that separation of constant is $\lambda$. The equation (5) becomes

$$\begin{aligned} X'' + \lambda X &= 0 \\ T'' + \lambda a^2 T &= 0 \end{aligned} \qquad \ldots (6)$$

And the condition (2) and (4) becomes

$$y(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0 \qquad ...(7)$$

$$\frac{\partial y}{\partial t}(x,0) = X(x)T'(0) = 0 \Rightarrow T'(0) = 0 \quad ...(8)$$

Now we will discuss the cases for different values of $\lambda$.

**Case 1:** If $\lambda = 0$

Then $X'' = 0 \quad \Rightarrow X(x) = Ax$

Which is unbounded solution on the given domain, unless $A = 0$. Thus, for this case we have a trivial solution.

**Case 2**: if $\lambda < 0$, let $\lambda = -p^2$ with $p > 0$.

Then $X'' - p^2 X = 0$ with the solution

$$X(x) = Ae^{px} + Be^{-px}$$

Now, $X(0) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$

$$\therefore \quad X(x) = A\left(e^{px} - e^{-px}\right)$$

which is unbounded solution for $p>0$ unless $A=0$. Thus $X(x)=0$, again we get a trivial solution.

**Case 3**: if $\lambda>0$, let $\lambda=p^2$ with $p>0$ .

Then $X''+p^2X=0$ with the solution

$$X(x)=A\cos px+B\sin px$$

Now,
$$X(0)=0 \Rightarrow C=0$$
$$\therefore \quad X(x)=D\sin px$$

Thus for each $p>0$, $\lambda=p^2$ is an eigen value and $X_p(x)=D_p\sin px$.

Now the problem for T is

$$T''+a^2p^2T=0 \text{ with the solution}$$
$$T(t)=C\cos(pat)+D\sin(pat)$$

Now,
$$T'(0)=0 \Rightarrow paD=0 \Rightarrow D=0$$
$$T(t)=C\cos(pat)$$

Thus for $p>0,\quad T_p(t)=C_p\cos\left(pat\right)$

Therefore, for this case

$$y_p(x,t)=E_p\sin\left(px\right)\cos\left(pat\right) \quad where\ E_p=A_pC_p$$

Using the superposition, we have

$$y(x,t)=\int_0^\infty E_p\sin\left(px\right)\cos\left(pat\right)dp$$

Also it is given $y(x,0)=f(x)\Rightarrow\int_0^\infty E_p\sin\left(px\right)dp=f(x)$

So
$$E_p=\frac{2}{\pi}\int_0^\infty f(\xi)\sin\left(p\xi\right)d\xi$$

**(i) Zero initial displacement**

For this case $f(x)=0$. For a bounded solution, we firstly set $y(x,t)=X(x)T(t)$.

Using this in equation, we get

$$\frac{X''}{X}=\frac{T''}{a^2T}$$

In equation, the left side is a function of $x$ only while right side is function of $t$. So each side must be equal to some constant, let that separation of constant is $\lambda$. The equation becomes

$$X'' + \lambda X = 0$$
$$T'' + \lambda a^2 T = 0$$

And the condition (2) and (4) becomes

$$y(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$$
$$y(x,0) = X(x)T(0) = 0 \Rightarrow T(0) = 0$$

Now we will discuss the cases for different values of $\lambda$.

**Case 1:** If $\lambda = 0$

Then $X'' = 0 \quad \Rightarrow X(x) = Ax$

Which is unbounded solution on the given domain, unless $A = 0$. Thus, for this case we have a trivial solution.

**Case 2**: if $\lambda < 0$, let $\lambda = -p^2$ with $p > 0$ .

Then $X'' - p^2 X = 0$ with the solution

$$X(x) = Ae^{px} + Be^{-px}$$

Now,
$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$$
$$\therefore \quad X(x) = A\left(e^{px} - e^{-px}\right)$$

Which is unbounded solution for $p > 0$ unless $A = 0$. Thus $X(x) = 0$, again we get a trivial solution.

**Case 3**: if $\lambda > 0$, let $\lambda = p^2$ with $p > 0$ .

Then $X'' + p^2 X = 0$ with the solution

$$X(x) = A\cos px + B\sin px$$

Now,
$$X(0) = 0 \Rightarrow C = 0$$
$$\therefore \quad X(x) = D\sin px$$

Thus for each $p > 0$, $\lambda = p^2$ is an eigen value and $X_p(x) = D_p \sin px$.

Now the problem for T is

$$T'' + a^2 p^2 T = 0 \text{ with the solution}$$
$$.T(t) = C\cos(pat) + D\sin(pat).$$

Now,
$$T(0) = 0 \Rightarrow paC = 0 \Rightarrow C = 0$$
$$T(t) = D\sin(pat)$$

Thus for $p > 0, \quad T_p(t) = D_p \sin\left(pat\right)$

Therefore, for this case

$$y_p(x,t) = E_p \sin(px)\sin(pat) \quad where\ E_p = A_p D_p$$

Using the superposition, we have

$$y(x,t) = \int_0^\infty E_p \sin(px)\sin(pat)\,dp$$

$$\frac{\partial y(x,0)}{\partial t} = g(x) \Rightarrow \int_0^\infty paE_p \sin(px)\,dp = g(x)$$

Also it is given

$$E_p = \frac{2}{pa\pi}\int_0^\infty f(\xi)\sin(p\xi)\,d\xi$$

Thus, the general solution is

$$y(x,t) = \frac{2}{a\pi}\int_0^\infty \left[\frac{1}{p}\int_0^\infty f(\xi)\sin(p\xi)\,d\xi\right]\sin(px)\cos(pat)\,dp$$

### 1.7.2 Wave Motion for a Infinite String

Let us consider an elastic string which stretched over a real line. The B.V.P. for the motion of infinite string is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad (-\infty < x < \infty,\ t > 0) \qquad ...(1)$$

$$y(x,0) = f(x), \qquad (-\infty < x < \infty) \qquad ...(2)$$

$$\frac{\partial y(x,0)}{\partial t} = g(x), \qquad (-\infty < x < \infty) \qquad ...(3)$$

Similar to previous article, we will separate the problem in two parts: (i) zero initial velocity and with displacement (ii) with initial velocity and zero displacement.

**Case (i) Zero initial velocity**

For this case $g(x) = 0$. For a bounded solution, we firstly set $y(x,t) = X(x)T(t)$. Using this in equation, we get

$$\frac{X''}{X} = \frac{T''}{a^2 T}$$

In equation, the left side is a function of $x$ only while right side is function of $t$. So each side must be equal to some constant, let that separation of constant is $\lambda$. The equation becomes

$$X'' + \lambda X = 0$$

$$T'' + \lambda a^2 T = 0$$

and the condition becomes

$$\frac{\partial y}{\partial t}(x,0) = X(x)T'(0) = 0 \;\Rightarrow T'(0) = 0$$

Now we will discuss the cases for different values of $\lambda$ .

**Case 1:** If $\lambda = 0$

$$\text{Then } X'' = 0 \quad \Rightarrow X(x) = Ax + B$$

Which is unbounded solution on the given domain, unless $A = 0$ . Thus solution is $X(x) = B$ for the eigen value.

**Case 2**: if $\lambda < 0$ , let $\lambda = -p^2$ with $p > 0$ .

Then $X'' - p^2 X = 0$ with the solution

$$X(x) = Ae^{px} + Be^{-px}$$

Since $p > 0$, the first term in right hand side $Ae^{px}$ is unbounded in the domain $[0,\infty)$ and the second term $Be^{-px}$ is unbounded in the region $(-\infty,0)$ Therefore, for a bounded solution, we have to assume that $A = 0$ *and* $B = 0$. Therefore $X(x) = 0$

**Case 3**: if $\lambda > 0$ , let $\lambda = p^2$ with $p > 0$ .

Then $X'' + p^2 X = 0$ with the solution

$$X(x) = A\cos px + B\sin px$$

The function $X(x)$ is always bounded for every $p > 0$ an so, we have

$$X_p(x) = A_p\cos(px) + B_p\sin(px)$$

Now the problem for T is

$T'' + \lambda a^2 T = 0$ and

$$\frac{\partial y}{\partial t}(x,0) = X(x)T'(0) = 0 \;\Rightarrow T'(0) = 0$$

If $\lambda = 0$, then we have
$$T(t) = Ct + D$$
$$\textit{and} \quad T'(0) = 0 \;\Rightarrow C = 0$$
$$\therefore\; T(t) = D$$

Is a solution for $T$ . On the other hand, if $\lambda = p^2,\; p > 0$ , then the equation $T'' + a^2 p^2 T = 0$ has the solution

$$T(t) = E\cos(pat) + F\sin(pat)$$

But
$$T'(0) = 0 \Rightarrow paF = 0 \Rightarrow F = 0$$
$$\therefore \quad T_p(t) = E_p \cos\left(pat\right)$$

Therefore, for this case

$$y_p(x,t) = \left[a_p \cos\left(px\right) + b_p \sin\left(px\right)\right]\cos\left(pat\right) \quad where\ a_p = A_p E_p \quad and \quad b_p = B_p E_p$$

Using the superposition, we have

$$y(x,t) = \int_{-\infty}^{\infty} \left[a_p \cos\left(px\right) + b_p \sin\left(px\right)\right]\cos\left(pat\right) dp$$

Also it is given $y(x,0) = f(x) = \int_{-\infty}^{\infty} \left[a_p \cos\left(px\right) + b_p \sin\left(px\right)\right]\cos\left(pat\right) dp$

Where
$$a_p = \frac{1}{\pi}\int_{-\infty}^{\infty} f\left(\xi\right)\cos\left(p\xi\right) dp$$
$$b_{p=} \frac{1}{\pi}\int_{-\infty}^{\infty} f\left(\xi\right)\sin\left(p\xi\right) dp$$

So $E_p = \frac{2}{\pi}\int_{0}^{\infty} f(\xi)\sin\left(p\xi\right) d\xi$

(ii) **Zero initial displacement**

For this case $f(x) = 0$. Similar to previous case, the eigen function for X is

$X_p(x) = A_p \cos\left(px\right) + A_p\, s in\left(px\right)$ with eigen values $\lambda = p^2$ with $p > 0$. And the solution for T

$$T_p(t) = E_p \cos\left(pat\right) + F_p \sin\left(pat\right)$$

The problem is same as zero initial velocity except the condition $y(x,0) = 0$.This implies $X(x)T(0) = 0 \Rightarrow T(0) = 0$. We have $E_p = 0$. The solution becomes $T_p(t) = F_p \sin\left(pat\right)$

Therefore, for this case, the solution is

$$y_p(x,t) = \left[a_p \cos\left(px\right) + b_p\, s in\left(px\right)\right]\sin\left(pat\right) \quad where\ a_p = A_p F_p \ \ and\ \ b_p = B_p F_p$$

Using the superposition, we have

$$y(x,t) = \int_{-\infty}^{\infty} \left[a_p \cos\left(px\right) + b_p\, s in\left(px\right)\right]\sin\left(pat\right) dp$$

Also it is given $\frac{\partial y(x,0)}{\partial t} = g(x) \Rightarrow \int_{-\infty}^{\infty} pa\left[a_p \cos\left(px\right) + b_p\, s in\left(px\right)\right]\sin\left(pat\right) dp = g(x)$ .

The coefficient $a_p$ and $b_p$ are given by

$$a_p = \frac{1}{ap\pi}\int_{-\infty}^{\infty} g(\xi)\cos(p\xi)\,d\xi$$
$$b_p = \frac{1}{ap\pi}\int_{-\infty}^{\infty} g(\xi)\sin(p\xi)\,d\xi$$

**Problems:**

**1.** Find the solution of B.V.P

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L,\ t > 0)$$
$$y(0,t) = y(L,t) = 0 \quad (t > 0)$$
$$y(x,0) = \begin{cases} x & ,0 \le x \le \dfrac{L}{2} \\ L - x & ,\dfrac{L}{2} \le x \le L \end{cases}$$
$$\frac{\partial y}{\partial t}(x,0) = x\left(\cos\left(\frac{\pi x}{L}\right)\right) \qquad (0 < x < L)$$

2. Find the solution of B.V.P.

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad (x > 0,\ t > 0)$$
$$y(0,t) = 0 \qquad (t > 0)$$
$$y(x,0) = \begin{cases} x(1-x) & ,0 < x < 1 \\ 0 & ,x > 1 \end{cases}$$
$$\frac{\partial y}{\partial t}(x,0) = 0 \qquad (x > 0)$$

**Some other problems**

**The Heat Equation in an Infinite Cylinder**

Suppose we want the temperature distribution in a solid, infinitely long, homogeneous circular cylinder of radius R. Let the z-axis be along the axis of the cylinder. In cylindrical co-ordinates the Heat equation is:

$$\frac{\partial u}{\partial t} = a^2\nabla^2 u = a^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

We assume that the temperature at any point in the cylinder depends only on the time t and the distance r from the z-axis, the axis of the cylinder. This means that $\dfrac{\partial u}{\partial \theta} = \dfrac{\partial u}{\partial z} = 0$ and the Heat equation is

$$V_n(r,\theta,\phi) = (Ar^n + Br^{-n-1})\Theta(\cos\theta)e^{\pm i\lambda\phi}$$

$$\frac{\partial u}{\partial t} = a^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]$$

Here u is a function of r and t only. The boundary condition we will consider is

$$u(R,t) = 0$$

for $t > 0$. This means that the outer surface is kept at temperature zero.

Now as in previous articles (left as an exercise for readers) we obtain,

$$u(r,t) = \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{z_n r}{R}\right)}{J_1^2(z_n)}e^{\frac{-a^2 z_n^2 t}{R^2}}\int_0^R f(\xi)\xi J_0\left(\frac{z_n\xi}{R}\right)d\xi$$

**Solve the following exercise:**

**Exercise:** A homogeneous circular cylinder of radius 2 and semi-infinite length has its base, which is sitting on the plane $z = 0$, maintained at a constant positive temperature K. The lateral surface is kept at temperature zero. Determine the steady-state temperature of the cylinder if it has a thermal diffusivity of $a^2$, assuming that the temperature at any point depends only on the height $z$ above the base and the distance $\mu$ from the axis of the cylinder.

**The Heat Equation in a Solid Sphere:**

Consider a solid sphere of radius R centered at the origin. We want to solve for the steady-state temperature distribution, given the temperature at all times on the surface

**Solution:** Here, it is natural to use spherical co-ordinates. We assume that temperature depends only on distance from the origin R. The angle of declination from the z-axis $\phi$, with $\frac{\partial u}{\partial \theta} = 0$, Laplace equation in spherical co-ordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad ...(1)$$

for $(0 < r \le R, 0 \le \phi \le \pi)$

The temperature is given on the surface

$$u(R,\phi) = f(\phi) \quad (0 \le \phi \le \pi) \qquad ...(2)$$

To solve this BVP, we set

$$u(r,\phi) = R(r)\Theta(\phi) \qquad ...(3)$$

(Remaining solution is left for readers as an exercise).